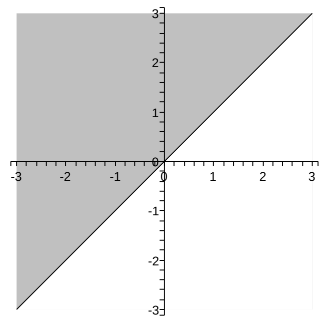
1. Consider the function

$$f(x, y) = \sqrt{y - x}.$$

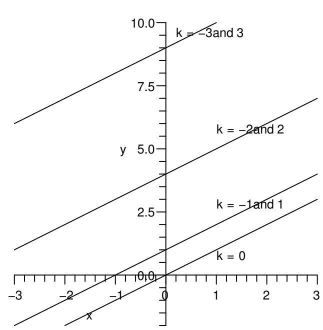
(a) Sketch the domain of f.

For the domain, need $y - x \geq 0$, i.e., $y \geq x$.



(b) Sketch the level curves $f(x, y) = k$ for $k = 0, 1, 2, 3$.

The level curves are $\sqrt{y - x} = k \Longrightarrow y - x = k^2 \Longrightarrow y = k^2 + x$. These are lines of slope 1 with k^2 being the y-intercept. Thus we have for $k = 0, \pm 1, \pm 2, \pm 3$:



(c) Find f_x.

$$f = (y - x)^{1/2} \Longrightarrow f_x = \frac{1}{2}(y - x)^{-1/2}(-1) = \boxed{\frac{-1}{2\sqrt{y - x}}}$$

(d) Find f_y.

$$f = (y - x)^{1/2} \Longrightarrow f_y = \frac{1}{2}(y - x)^{-1/2}(1) = \boxed{\frac{1}{2\sqrt{y - x}}}$$

(e) Find the equation of the tangent plane to f at the point $(2, 6, 2)$.

$$\text{Equation: } z - z_0 = [f_x(x_0, y_0)](x - x_0) + [f_y(x_0, y_0)](y - y_0)$$

$$f_x(x_0, y_0) = f_x(2, 6) = \frac{-1}{2\sqrt{6 - 2}} = -\frac{1}{4}, \qquad f_y(x_0, y_0) = f_y(2, 6) = \frac{1}{2\sqrt{6 - 2}} = \frac{1}{4}$$

$$\boxed{z - 2 = -\frac{1}{4}(x - 2) + \frac{1}{4}(y - 6)}$$

2. Show that $\lim_{(x,y)\to(0,0)} \frac{3xy}{x^2+y^2}$ doesn't exist.

$$\text{When } x=0, y\to 0 \qquad \lim_{(x,y)\to(0,0)} \frac{3xy}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{0}{y^2} = 0$$
$$\text{When } x=y, \qquad \lim_{(x,y)\to(0,0)} \frac{3xy}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{3x^2}{2x^2} = \frac{3}{2}$$

Since we approach $(0,0)$ from 2 different directions we then get 2 different values, the limit doesn't exist.

3. Given $u(t,x) = e^{-\alpha^2 k^2 t}\sin(kx)$, evaluate $\frac{\partial u}{\partial t} - \alpha^2 \frac{\partial^2 u}{\partial x^2}$, simplifying as much as possible.

$$\begin{aligned}
\frac{\partial u}{\partial t} &= -\alpha^2 k^2 e^{-\alpha^2 k^2 t}\sin(kx) \\
\frac{\partial u}{\partial x} &= k e^{-\alpha^2 k^2 t}\cos(kx) \\
\frac{\partial^2 u}{\partial x^2} &= -k^2 e^{-\alpha^2 k^2 t}\sin(kx) \\
\therefore \frac{\partial u}{\partial t} - \alpha^2 \frac{\partial^2 u}{\partial x^2} &= \boxed{0}
\end{aligned}$$

4. Given $x - z = \arctan(yz)$, find

(a) $\frac{\partial z}{\partial x}$

$$\begin{aligned}
1 - \frac{\partial z}{\partial x} &= \frac{1}{1+(yz)^2}\left(y\frac{\partial z}{\partial x}\right) \\
1 &= \frac{y}{1+y^2z^2}\frac{\partial z}{\partial x} + \frac{\partial z}{\partial x} \\
1 &= \left(1 + \frac{y}{1+y^2z^2}\right)\frac{\partial z}{\partial x} \\
1 &= \left(\frac{1+y^2z^2+y}{1+y^2z^2}\right)\frac{\partial z}{\partial x} \\
\frac{\partial z}{\partial x} &= \boxed{\frac{1+y^2z^2}{1+y^2z^2+y}}
\end{aligned}$$

OR, use the Chain Rule in §14.5:

$$\begin{aligned}
F(x,y,z) &= x - z - \arctan(yz) = 0 \\
\frac{\partial z}{\partial x} &= \frac{-F_x}{F_z} \\
&= \frac{-1}{-1 - \frac{y}{1+(yz)^2}} \qquad \text{(factor out } -1 \text{ from top and bottom...)} \\
&= \frac{1}{\frac{1+y^2z^2+y}{1+y^2z^2}} = \boxed{\frac{1+y^2z^2}{1+y^2z^2+y}}
\end{aligned}$$

(b) $\dfrac{\partial z}{\partial y}$

$$\begin{aligned}
-\frac{\partial z}{\partial y} &= \frac{1}{1+y^2z^2}\left(z + y\frac{\partial z}{\partial y}\right) \\
\frac{-z}{1+y^2z^2} &= \frac{\partial z}{\partial y} + \frac{y}{1+y^2z^2}\frac{\partial z}{\partial y} \\
\frac{-z}{1+y^2z^2} &= \frac{\partial z}{\partial y}\left(1 + \frac{y}{1+y^2z^2}\right) = \frac{\partial z}{\partial y}\left(\frac{1+y^2z^2+y}{1+y^2z^2}\right) \\
\frac{\partial z}{\partial y} &= \boxed{\frac{-z}{1+y^2z^2+y}}
\end{aligned}$$

OR, use Chain Rule in §14.5:

$$\begin{aligned}
F(x,y,z) &= x - z - \arctan(yz) = 0 \\
\frac{\partial z}{\partial y} &= \frac{-F_y}{F_z} \\
&= \frac{-\left(\frac{-z}{1+(yz)^2}\right)}{-1 - \frac{y}{1+(yz)^2}} \\
&= \frac{\frac{z}{1+y^2z^2}}{\frac{-(1+y^2z^2+y)}{1+y^2z^2}} = \boxed{\frac{-z}{1+y^2z^2+y}}
\end{aligned}$$

5. Consider the surface given implicitly by $xy + yz + xz = 7$. Find

(a) $\dfrac{\partial z}{\partial x}$

$$\begin{aligned}
y + y\frac{\partial z}{\partial x} + z + x\frac{\partial z}{\partial x} &= 0 \\
\frac{\partial z}{\partial x}(x+y) &= -y - z \\
\frac{\partial z}{\partial x} &= \boxed{\frac{-(y+z)}{x+y}}
\end{aligned}$$

OR, use Chain Rule in §14.5:

$$\begin{aligned}
F(x,y,z) &= xy + yz + xz - 7 = 0 \\
\frac{\partial z}{\partial x} &= \frac{-F_x}{F_z} \\
&= \boxed{\frac{-(y+z)}{y+x}}
\end{aligned}$$

(b) $\dfrac{\partial z}{\partial y}$

$$\begin{aligned}
x + z + y\frac{\partial z}{\partial y} + x\frac{\partial z}{\partial y} &= 0 \\
\frac{\partial z}{\partial y}(x+y) &= -x - z \\
\frac{\partial z}{\partial y} &= \boxed{\frac{-(x+z)}{x+y}}
\end{aligned}$$

OR, use Chain Rule in §14.5:

$$\begin{aligned}
F(x,y,z) &= xy + yz + xz - 7 = 0 \\
\frac{\partial z}{\partial y} &= \frac{-F_y}{F_z} \\
&= \boxed{\frac{-(x+z)}{y+x}}
\end{aligned}$$

6. Suppose $u = xy + yz + xz$, $x = st$, $y = e^{st}$ and $z = t^2$.

   (a) Find $\dfrac{\partial u}{\partial s}$ at the point $(s, t) = (0, 1)$.

$$\begin{aligned}
\frac{\partial u}{\partial s} &= \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial s} \\
&= (y+z)t + (x+z)te^{st} + (y+x)0 \\
(s,t) = (0,1) &\implies x = 0,\ y = 1,\ z = 1 \\
\left.\frac{\partial u}{\partial s}\right|_{(s,t)=(0,1)} &= 2 + 1 = \boxed{3}
\end{aligned}$$

   (b) Find $\dfrac{\partial u}{\partial t}$ at the point $(s, t) = (0, 1)$.

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial t} \\
&= (y+z)s + (x+z)se^{st} + (y+x)2t \\
\left.\frac{\partial u}{\partial t}\right|_{(s,t)=(0,1)} &= 1 \cdot 0 + 1 \cdot 0 + 1 \cdot 2 = \boxed{2}
\end{aligned}$$

7. Suppose $z = 2xy + 3y^2 + ye^x$, $x = r\cos\theta$ and $y = r\sin\theta$.

   (a) Find $\dfrac{\partial z}{\partial r}$ at the point $(r, \theta) = (2, \pi)$.

$$\begin{aligned}
\frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \\
&= (2y + ye^x)(\cos\theta) + (2x + 6y + e^x)(\sin\theta) \\
(r,\theta) = (2,\pi) &\implies x = -2,\ y = 0 \\
\left.\frac{\partial z}{\partial r}\right|_{(r,\theta)=(2,\pi)} &= (0)(-1) + (-4 + e^{-2})(0)
\end{aligned}$$

$$\boxed{\left.\frac{\partial z}{\partial r}\right|_{(r,\theta)=(2,\pi)} = 0}$$

   (b) Find $\dfrac{\partial z}{\partial \theta}$ at the point $(r, \theta) = (2, \pi)$.

$$\begin{aligned}
\frac{\partial z}{\partial \theta} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} \\
&= (2y + ye^x)(-r\sin\theta) + (2x + 6y + e^x)(r\cos\theta) \\
(r,\theta) = (2,\pi) &\implies x = -2,\ y = 0 \\
\left.\frac{\partial z}{\partial \theta}\right|_{(r,\theta)=(2,\pi)} &= (0)(0) + (-4 + e^{-2})(-2)
\end{aligned}$$

$$\boxed{\left.\frac{\partial z}{\partial \theta}\right|_{(r,\theta)=(2,\pi)} = 8 - 2e^{-2}}$$

8. Verify that the function $f(x,y) = \ln\sqrt{x^2+y^2}$ is a solution to Laplace's equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

$$\begin{aligned}
f(x,y) &= \ln(x^2+y^2)^{1/2} \\
\frac{\partial f}{\partial x} &= \frac{1}{\sqrt{x^2+y^2}}\left(\frac{1}{2}(x^2+y^2)^{-1/2}(2x)\right) = \frac{x}{x^2+y^2} \\
\frac{\partial^2 f}{\partial x^2} &= \frac{(x^2+y^2)-x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \\
\frac{\partial f}{\partial y} &= \frac{1}{\sqrt{x^2+y^2}}\left(\frac{1}{2}(x^2+y^2)^{-1/2}(2y)\right) = \frac{y}{x^2+y^2} \\
\frac{\partial^2 f}{\partial y^2} &= \frac{(x^2+y^2)-y(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2} \\
\therefore \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} &= \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} = 0
\end{aligned}$$

9. Calculate the limits, or show that they don't exist.

(a) $\displaystyle\lim_{(x,y,z)\to(4,1,-2)} e^{x^2 z}\cos(2y+z)$

$$\begin{aligned}
\lim_{(x,y,z)\to(4,1,-2)} e^{x^2 z}\cos(2y+z) &= e^{16\cdot(-2)}\cos(2-2) \\
&= \boxed{e^{-32}}
\end{aligned}$$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{5x^4y^2}{x^8+y^8}$

If we let $x = 0$ and $y \to 0$, $f(x,y) = \dfrac{0}{y^8} \to 0$.

If we let $x = y$ and let $(x,y) \to (0,0)$, $f(x,y) = \dfrac{5x^6}{2x^8} = \dfrac{5}{2x^2} \to \infty$. Since when we approach $(0,0)$ from 2 different directions we get to different limits, the limit does not exist.

10. Find all second partial derivatives of $f(x,y) = \ln(3x+5y)$.

$$\begin{aligned}
f_x &= \frac{3}{3x+5y} = 3(3x+5y)^{-1} \\
f_{xx} &= -3(3x+5y)^{-2}(3) \\
\Longrightarrow & \boxed{f_{xx} = \frac{-9}{(3x+5y)^2}} \\
f_{xy} &= -3(3x+5y)^{-2}(5) \\
\Longrightarrow & \boxed{f_{xy} = \frac{-15}{(3x+5y)^2}}
\end{aligned}$$

$$\begin{aligned}
f_y &= \frac{5}{3x+5y} = 5(3x+5y)^{-1} \\
f_{yy} &= -5(3x+5y)^{-2}(5) \\
\Longrightarrow & \boxed{f_{yy} = \frac{-25}{(3x+5y)^2}} \\
f_{yx} &= -5(3x+5y)^{-2}(3) \\
\Longrightarrow & \boxed{f_{yx} = \frac{-15}{(3x+5y)^2}}
\end{aligned}$$

11. Find the linear approximation of $f(x,y) = \ln(x-3y)$ at $(7,2)$ and use it to approximate $f(6.9, 2.06)$.

$$\begin{aligned}
L(x,y) &= f_x(7,2)(x-7) + f_y(7,2)(y-2) + f(7,2) \\
f_x &= \frac{1}{x-3y} \Longrightarrow f_x(7,2) = \frac{1}{7-6} = 1 \\
f_y &= \frac{-3}{x-3y} \Longrightarrow f_y(7,2) = \frac{-3}{7-6} = -3 \\
f(7,2) &= \ln(1) = 0 \\
L(x,y) &= \boxed{(x-7) + (-3)(y-2) = x - 7 - 3y + 6 = x - 3y - 1} \\
L(6.9, 2.06) &= -0.1 + (-3)(0.06) = -.1 - .18 = \boxed{-.28 \approx f(6.9, 2.06)}
\end{aligned}$$

12. The pressure, volume and temperature of a mole of an ideal gas are related by teh equation $PV = 8.31T$, where P is measured in kilopascals, V in liters, and T in kelvins. Use differentials to find the approximate change in the pressure if the volume increases from 12 L to 12.3 L and the temperature decreases from 310 K to 305 K.

$$\begin{aligned}
P &= 8.31TV^{-1} \\
dP &= \frac{\partial P}{\partial T}dT + \frac{\partial P}{\partial V}dV, \\
&= 8.31V^{-1}dT + -8.31TV^{-2}dV \\
V = 12,\ T = 310,&\ dT = -5,\ dV = 0.3 \\
dP &= \frac{-41.55}{12} + \frac{-772.83}{144} \approx -8.83
\end{aligned}$$

Thus the pressure will drop by about 8.83 kPa.

13. Consider the function $f(x,y) = 3x^2 - xy + y^3$.

(a) Find the rate of change of f at $(1,2)$ in the direction of $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}$.

$$\begin{aligned}
\text{use } \mathbf{v} = \mathbf{u}/\|\mathbf{u}\| &= < 3/5,\ 4/5 > \\
\nabla f &= < 6x - y,\ -x + 3y^2 > \\
\nabla f(1,2) &= < 4,\ 11 > \\
D_{\mathbf{u}} f(1,2) = \nabla f(1,2) \cdot \mathbf{v} &= 12/5 + 44/5 \\
&= \boxed{\frac{56}{5}}
\end{aligned}$$

(b) From the point $(1,2)$, in what direction does f decrease the most? Give your answer as a unit vector. What is this maximum rate of decrease?

Decreases the most in the direction of $-\nabla f$ so in the direction of $< -4, -11 >$. Answer needs to be a unit vector, norm is $\sqrt{16+121} = \sqrt{137}$

$$\boxed{\text{Decreases the most in the direction of } \left\langle -\frac{4}{\sqrt{137}}, -\frac{11}{\sqrt{137}} \right\rangle \text{ with rate of decrease of } \sqrt{137}}$$

(c) From the point $(1, 2)$, in what direction does f increase the most? Give your answer as a unit vector. What is this maximum rate of increase?

Increases the most in the direction of ∇f

$$\boxed{\text{Increases the most in the direction of } \left\langle \frac{4}{\sqrt{137}}, \frac{11}{\sqrt{137}} \right\rangle \text{ with rate of decrease of } \sqrt{137}}$$

(d) From the point $(1, 2)$, in what direction(s) is the rate of change of f equal to zero? Give your answer(s) as unit vector(s).

Need $\nabla f(1, 2) \cdot \mathbf{v} = 0$.

$$\boxed{\left\langle \frac{11}{\sqrt{137}}, -\frac{4}{\sqrt{137}} \right\rangle \text{ and } \left\langle -\frac{11}{\sqrt{137}}, \frac{4}{\sqrt{137}} \right\rangle}$$

14. Suppose $f(x, y)$ is a function such that $\nabla f(2, 4)$ has norm of 5. Is there a direction $\mathbf{u}$ such that the directional derivative $D_{\mathbf{u}} f(2, 4) = 7$? Explain your answer.

No. Because the maximum the directional derivative can be is $\|\nabla f(2, 4)\| = 5$.

15. Consider the ellipsoid $x^2 + 4y^2 = 169 - 9z^2$ and the point $P(3, 2, 4)$ on the ellipsoid.

(a) Find the equation of the tangent plane to the ellipsoid at the point P.

$$\begin{aligned} F(x, y, z) &= x^2 + 4y^2 + 9z^2 - 169 = 0 \\ F_x &= 2x, \quad F_x(3, 2, 4) = 6 \\ F_y &= 8y, \quad F_y(3, 2, 4) = 16 \\ F_z &= 18z \quad F_z(3, 2, 4) = 72 \end{aligned}$$

$$\boxed{6(x - 3) + 16(y - 2) + 72(z - 4) = 0}$$

(b) Find the parametric equations for the normal line to the ellipsoid at the point P.

Use $< 6, 16, 72 >$ as the direction vector and the point $(3, 2, 4)$.

$$\boxed{\begin{aligned} x &= 6t + 3 \\ y &= 16t + 2 \\ z &= 72t + 4 \end{aligned}}$$

16. Let $f(x, y) = x^2 + \dfrac{y^2}{2} + x^2 y$.

(a) Find all critical points of f.

$$\begin{aligned} f_x &= 2x + 2xy \\ &= 2x(1 + y) = 0 \text{ when } x = 0 \text{ or } y = -1 \\ f_y &= y + x^2 \\ x = 0 \Longrightarrow f_y &= y = 0 \text{ when } y = 0 \\ y = -1 \Longrightarrow f_y &= -1 + x^2 = 0 \text{ when } x = \pm 1 \end{aligned}$$

$$\boxed{(0, 0), \quad (1, -1), \quad (-1, -1)}$$

(b) Apply the second derivative test to each of them, and write down the result of the test.

$$f_{xx} = 2 + 2y \qquad f_{xy} = f_{yx} = 2x \qquad f_{yy} = 1$$

$$D(0,0) = \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = 2 \qquad D(0,0) > 0, \quad f_{xx}(0,0) > 0 \Longrightarrow \text{ local min}$$

$$D(1,-1) = \begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix} = -4 \qquad D(1,-1) < 0 \Longrightarrow \text{ saddle point}$$

$$D(-1,-1) = \begin{vmatrix} 0 & -2 \\ -2 & 1 \end{vmatrix} = -4 \qquad D(-1,-1) < 0 \Longrightarrow \text{ saddle point}$$

$$\boxed{(0,0) \text{ is a local min, } (1,-1) \text{ and } (-1,-1) \text{ are saddle points}}$$

17. Consider the function $f(x, y) = y^2 - x^2 + x^4$. Find and classify (as maxima, minima or saddles) the critical points of f, showing all work.

$$f_x = -2x + 4x^3 = -2x(1 - 2x^2) = 0 \quad \text{when } x = 0,\ x = \pm\frac{1}{\sqrt{2}}$$

$$f_y = 2y = 0 \quad \text{when } y = 0$$

$$f_{xx} = -2 + 12x^2 \qquad f_{xy} = f_{yx} = 0 \qquad f_{yy} = 2$$

$$D(0,0) = \begin{vmatrix} -2 & 0 \\ 0 & 2 \end{vmatrix} = -4 \qquad D\left(\frac{1}{\sqrt{2}}, 0\right) = \begin{vmatrix} 4 & 0 \\ 0 & 2 \end{vmatrix} = 8 \qquad D\left(-\frac{1}{\sqrt{2}}, 0\right) = \begin{vmatrix} 4 & 0 \\ 0 & 2 \end{vmatrix} = 8$$

$$f_{xx}\left(\pm\frac{1}{\sqrt{2}}, 0\right) > 0$$

$$\boxed{(0,0) \text{ is a saddlepoint, } \left(\pm\frac{1}{\sqrt{2}}, 0\right) \text{ are local minimums}}$$

18. Find the maximum of $f(x, y) = xy$ restricted to the curve $(x+1)^2 + y^2 = 1$. Give both the coordinates of the point and the value of f.

$$\begin{aligned} f(x,y) &= xy \\ g(x,y) &= (x+1)^2 + y^2 = 1 \\ \nabla f &= < y,\ x > \\ \nabla g &= < 2(x+1),\ 2y > \end{aligned}$$

$$\begin{aligned} \Longrightarrow \text{ solve } y &= \lambda(2x+2) && (1) \\ x &= \lambda 2y && (2) \\ (x+1)^2 + y^2 &= 1 && (3) \end{aligned}$$

$$(2) \Longrightarrow \lambda = \frac{x}{2y} \text{ OR } x = y = 0$$

$$\lambda = \frac{x}{2y} \overset{(1)}{\Longrightarrow} y = \frac{x}{2y}(2x+2) = \frac{2x^2 + 2x}{2y}$$

$$\Longrightarrow 2y^2 = 2x^2 + 2x \Longrightarrow y^2 = x^2 + x$$

$$\overset{(3)}{\Longrightarrow} x^2 + 2x + 1 + x^2 + x = 1$$

$$2x^2 + 3x = 0 \Longrightarrow x = 0,\ -\frac{3}{2}$$

$$x = 0 \Longrightarrow y = 0$$
$$x = -\frac{3}{2} \Longrightarrow y^2 = \frac{9}{4} - \frac{3}{2} = \frac{3}{4}$$
$$\Longrightarrow y = \pm\frac{\sqrt{3}}{2}$$

$$f(0,0) = 0 \qquad f\left(-\frac{3}{2}, -\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{4} \qquad f\left(-\frac{3}{2}, \frac{\sqrt{3}}{2}\right) = -\frac{3\sqrt{3}}{4}$$

$$\boxed{\text{maximum is } \frac{3\sqrt{3}}{4} \text{ at } \left(-\frac{3}{2}, -\frac{\sqrt{3}}{2}\right)}$$

19. Find the dimensions of a rectangular box of maximum volume such that the sum of the lengths of its 12 edges is a constant, C.

$$\text{dimensions } x, y, z \Longrightarrow C = 4x + 4y + 4z$$
$$z = \frac{C}{4} - x - y$$
$$\text{maximize } V = xyz = xy\left(\frac{C}{4} - x - y\right) = \frac{Cxy}{4} - x^2y - xy^2$$
$$\frac{\partial V}{\partial x} = \frac{Cy}{4} - 2xy - y^2 = y\left(\frac{C}{4} - 2x - y\right)$$
$$= 0 \text{ when } y = 0 \text{ or } y = \frac{C}{4} - 2x$$

$y = 0$ is not a realistic solution for this problem

$$\frac{\partial V}{\partial y} = \frac{Cx}{4} - x^2 - 2xy$$
$$y = \frac{C}{4} - 2x \Longrightarrow \frac{\partial V}{\partial y} = \frac{Cx}{4} - x^2 - 2x\left(\frac{C}{4} - 2x\right)$$
$$= \frac{Cx}{4} - x^2 - \frac{2Cx}{4} + 4x^2$$
$$= 3x^2 - \frac{Cx}{4} = 0 \text{ when } x = 0,\ x = \frac{C}{12}$$
$$x = \frac{C}{12} \Longrightarrow y = \frac{C}{12} = z$$
$$V_{xx} = -2y, \quad V_{xy} = \frac{C}{4} - 2x, \quad V_{yy} = -2x$$
$$D > 0, V_{xx} < 0 \Longrightarrow \text{ a maximum}$$

$$\boxed{\frac{C}{12} \times \frac{C}{12} \times \frac{C}{12}}$$

20. Find the absolute maximum and minimum of $f(x,y) = x^2 + xy + y^2$ over the disk $\{(x,y) \mid x^2 + y^2 \le 9\}$.

$$\text{Interior: } f_x = 2x + y = 0 \Longrightarrow y = -2x$$
$$f_y = x + 2y = 0 \Longrightarrow x + 2(-2x) = 0$$
$$\Longrightarrow -3x = 0 \Longrightarrow x = 0,\ y = 0$$

only critical point is $(0,0)$, $f(0,0) = 0$

$$\begin{aligned} \text{Boundary: } f(x,y) &= x^2 + xy + y^2 \\ g(x,y) &= x^2 + y^2 = 9 \\ \nabla f &= < 2x+y,\ x+2y > \\ \nabla g &= < 2x, 2y > \end{aligned} \qquad \Longrightarrow \text{ solve } \begin{aligned} 2x + y &= \lambda 2x && (1) \\ x + 2y &= \lambda 2y && (2) \\ x^2 + y^2 &= 9 && (3) \end{aligned}$$

$$\begin{aligned} 2x + y = \lambda 2x &\Longrightarrow y = 2x(\lambda - 1) \\ \Longrightarrow (2): \quad x + 4x(\lambda-1) &= \lambda 2y = \lambda 4x(\lambda - 1) \\ x + 4x\lambda - 4x &= 4x\lambda^2 - 4x\lambda \\ 4x\lambda^2 - 8\lambda x + 3x &= 0 \\ x(2\lambda - 1)(2\lambda - 3) &= 0 \\ \Longrightarrow x = 0,\ \lambda &= \frac{1}{2},\ \frac{3}{2} \\ x = 0,\ (3) &\Longrightarrow y = \pm 3 \\ f(0, \pm 3) &= 9 \\ \lambda = \frac{1}{2} &\Longrightarrow y = 2x(-1/2) = -x \\ y = -x,\ &\overset{(3)}{\Longrightarrow} x = \pm \frac{3}{\sqrt{2}} \end{aligned}$$

$$\begin{aligned} f\left(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right) &= \frac{9}{2} - \frac{9}{2} + \frac{9}{2} = \frac{9}{2} \\ f\left(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) &= \frac{9}{2} - \frac{9}{2} + \frac{9}{2} = \frac{9}{2} \\ \lambda = \frac{3}{2} &\Longrightarrow y = 3x - 2x = x \\ x^2 + y^2 = 9 &\Longrightarrow x = \pm \frac{3}{\sqrt{2}} \\ f\left(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right) &= \frac{9}{2} + \frac{9}{2} + \frac{9}{2} = \frac{27}{2} \\ f\left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) &= \frac{9}{2} + \frac{9}{2} + \frac{9}{2} = \frac{27}{2} \end{aligned}$$

$$\boxed{\text{minimum is 0 at } (0,0),\ \text{maximum is } \frac{27}{2} \text{ at } \left(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right), \left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)}$$