1. Find the maximum of $f(x, y) = xy$ restricted to the curve $(x+1)^2+y^2 = 1$. Give both the coordinates of the point and the value of f.

$$f(x,y) = xy$$
$$g(x,y) = (x+1)^2 + y^2 = 1$$
$$\nabla f = < y,\ x >$$
$$\nabla g = < 2(x+1),\ 2y >$$

$$\Longrightarrow \text{solve } y = \lambda(2x+2) \quad (1)$$
$$x = \lambda 2y \quad (2)$$
$$(x+1)^2 + y^2 = 1 \quad (3)$$

$$(2) \Longrightarrow \lambda = \frac{x}{2y} \text{ OR } x = y = 0$$
$$\lambda = \frac{x}{2y} \overset{(1)}{\Longrightarrow} y = \frac{x}{2y}(2x+2) = \frac{2x^2+2x}{2y}$$
$$\Longrightarrow 2y^2 = 2x^2 + 2x \Longrightarrow y^2 = x^2 + x$$
$$\overset{(3)}{\Longrightarrow} x^2 + 2x + 1 + x^2 + x = 1$$
$$2x^2 + 3x = 0 \Longrightarrow x = 0,\ -\frac{3}{2}$$
$$x = 0 \Longrightarrow y = 0$$
$$x = -\frac{3}{2} \Longrightarrow y^2 = \frac{9}{4} - \frac{3}{2} = \frac{3}{4}$$
$$\Longrightarrow y = \pm\frac{\sqrt{3}}{2}$$
$$f(0,0) = 0 \qquad f\left(-\frac{3}{2},\ -\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{4} \qquad f\left(-\frac{3}{2},\ \frac{\sqrt{3}}{2}\right) = -\frac{3\sqrt{3}}{4}$$

$$\boxed{\text{maximum is } \frac{3\sqrt{3}}{4} \text{ at } \left(-\frac{3}{2},\ -\frac{\sqrt{3}}{2}\right)}$$

2. Find the dimensions of a rectangular box of maximum volume such that the sum of the lengths of its 12 edges is a constant, C.

$$\text{dimensions } x, y, z \Longrightarrow C = 4x + 4y + 4z$$
$$z = \frac{C}{4} - x - y$$
$$\text{maximize } V = xyz = xy\left(\frac{C}{4} - x - y\right) = \frac{Cxy}{4} - x^2y - xy^2$$
$$\frac{\partial V}{\partial x} = \frac{Cy}{4} - 2xy - y^2 = y\left(\frac{C}{4} - 2x - y\right)$$
$$= 0 \text{ when } y = 0 \text{ or } y = \frac{C}{4} - 2x$$

$y = 0$ is not a realistic solution for this problem

$$\begin{aligned}
\frac{\partial V}{\partial y} &= \frac{Cx}{4} - x^2 - 2xy \\
y = \frac{C}{4} - 2x \Longrightarrow \frac{\partial V}{\partial y} &= \frac{Cx}{4} - x^2 - 2x\left(\frac{C}{4} - 2x\right) \\
&= \frac{Cx}{4} - x^2 - \frac{2Cx}{4} + 4x^2 \\
&= 3x^2 - \frac{Cx}{4} = 0 \text{ when } x = 0,\ x = \frac{C}{12} \\
x = \frac{C}{12} &\Longrightarrow y = \frac{C}{12} = z \\
V_{xx} = -2y, \quad V_{xy} &= \frac{C}{4} - 2x, \quad V_{yy} = -2x \\
D > 0, V_{xx} < 0 &\Longrightarrow \text{ a maximum}
\end{aligned}$$

$$\boxed{\frac{C}{12} \times \frac{C}{12} \times \frac{C}{12}}$$

3. Find the absolute maximum and minimum of $f(x,y) = x^2 + xy + y^2$ over the disk $\{(x,y)\,|\,x^2+y^2 \le 9\}$ and where they occur.

$$\begin{aligned}
\text{Interior: } f_x &= 2x + y = 0 \Longrightarrow y = -2x \\
f_y &= x + 2y = 0 \Longrightarrow x + 2(-2x) = 0 \\
&\Longrightarrow -3x = 0 \Longrightarrow x = 0,\ y = 0 \\
\text{only critical point is } &(0,0), f(0,0) = 0
\end{aligned}$$

$$\begin{aligned}
\text{Boundary: } f(x,y) &= x^2 + xy + y^2 \\
g(x,y) &= x^2 + y^2 = 9 \\
\nabla f &= < 2x + y,\ x + 2y > \\
\nabla g &= < 2x, 2y >
\end{aligned}
\qquad
\begin{aligned}
\Longrightarrow \text{ solve } 2x + y &= \lambda 2x && (1) \\
x + 2y &= \lambda 2y && (2) \\
x^2 + y^2 &= 9 && (3)
\end{aligned}$$

$$\begin{aligned}
2x + y = \lambda 2x &\Longrightarrow y = 2x(\lambda - 1) \\
\Longrightarrow (2): \quad x + 4x(\lambda - 1) &= \lambda 2y = \lambda 4x(\lambda - 1) \\
x + 4x\lambda - 4x &= 4x\lambda^2 - 4x\lambda \\
4x\lambda^2 - 8\lambda x + 3x &= 0 \\
x(2\lambda - 1)(2\lambda - 3) &= 0 \\
\Longrightarrow x = 0,\ \lambda &= \frac{1}{2},\ \frac{3}{2} \\
x = 0,\ (3) &\Longrightarrow y = \pm 3 \\
f(0, \pm 3) &= 9 \\
\lambda = \frac{1}{2} &\Longrightarrow y = 2x(-1/2) = -x \\
y = -x, &\overset{(3)}{\Longrightarrow} x = \pm\frac{3}{\sqrt{2}}
\end{aligned}$$

$$f\left(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right) = \frac{9}{2} - \frac{9}{2} + \frac{9}{2} = \frac{9}{2}$$
$$f\left(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) = \frac{9}{2} - \frac{9}{2} + \frac{9}{2} = \frac{9}{2}$$
$$\lambda = \frac{3}{2} \Longrightarrow y = 3x - 2x = x$$
$$x^2 + y^2 = 9 \Longrightarrow x = \pm\frac{3}{\sqrt{2}}$$
$$f\left(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right) = \frac{9}{2} + \frac{9}{2} + \frac{9}{2} = \frac{27}{2}$$
$$f\left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right) = \frac{9}{2} + \frac{9}{2} + \frac{9}{2} = \frac{27}{2}$$

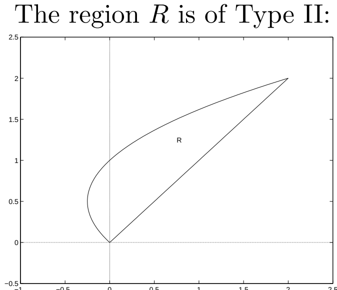
$$\boxed{\text{minimum is 0 at } (0,0)\text{, maximum is } \frac{27}{2} \text{ at } \left(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right), \left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)}$$

4. Find the volume of the solid lying under the circular paraboloid $z = x^2 + y^2$ and above the rectangle $R = [-2, 2] \times [-3, 3]$.

$$\begin{aligned}
\int_{-3}^{2}\int_{-2}^{2} x^2 + y^2 \, dx\, dy &= \int_{-3}^{3} \left[\frac{1}{3}x^3 + y^2 x\Big|_{x=-2}^{x=2}\right] dy \\
&= \int_{-3}^{3} \frac{8}{3} + 2y^2 - (-\frac{8}{3} - 2y^2)\, dy \\
&= \int_{-3}^{3} \frac{16}{3} + 4y^2 \, dy = \frac{16}{3}y + \frac{4}{3}y^3\Big|_{-3}^{3} \\
&= 16 + 36 - (-16 - 36) = \boxed{104}
\end{aligned}$$

5. Find the volume of the solid under the paraboloid $z = 3x^2 + y^2$ and above the region bounded by $y = x$ and $x = y^2 - y$.

The region R is of Type II:



$$\begin{aligned}
V &= \int_0^2 \int_{y^2-y}^{y} (3x^2 + y^2)\, dx\, dy \\
&= \int_0^2 \left[x^3 + xy^2\Big|_{x=y^2-y}^{x=y}\right] dy \\
&= \int_0^2 \left[y^3 + y^3 - \left((y^2 - y)^3 + (y^2 - y)y^2\right)\right] dy \\
&= \int_0^2 \left[2y^3 - \left(y^6 - 3y^5 + 3y^4 - y^3 + y^4 - y^3\right)\right] dy \\
&= \int_0^2 (4y^3 - 4y^4 + 3y^5 - y^6)\, dy \\
&= y^4 - \frac{4}{5}y^5 + \frac{1}{2}y^6 - \frac{1}{7}y^7\Big|_0^2 = \boxed{\frac{144}{35}}
\end{aligned}$$

6. Evaluate $\displaystyle\iint_D x\sqrt{y^2-x^2}\,dA$, $D=\{(x,y)\,|\,0\le y\le 1,\ 0\le x\le y\}$.

$$\iint_D x\sqrt{y^2-x^2}\,dA = \int_0^1\int_0^y x(y^2-x^2)^{\frac{1}{2}}\,dx\,dy \quad \text{(use u-sub on y^2-x^2)}$$

$$= \int_0^1\left[-\frac{1}{3}(y^2-x^2)^{3/2}\Big|_{x=0}^{x=y}\right]dy = \int_0^1\left[-\frac{1}{3}(y^2-y^2)^{3/2}+\frac{1}{3}(y^2)^{3/2}\right]dy$$

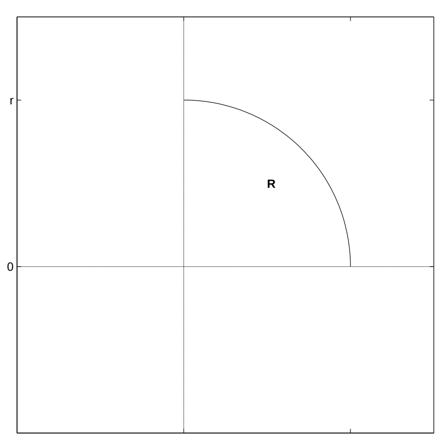
$$= \int_0^1 \frac{1}{12}y^3\,dy = \frac{1}{12}y^4\Big|_0^1 = \boxed{\frac{1}{12}}$$

7. Find the volume of the solid under the paraboloid $z=x^2+y^2+4$ and the planes $x=0$, $y=0$, $z=0$ and $x+y=1$.

$$\begin{aligned}
V &= \int_0^1\int_0^{1-x}(x^2+y^2+4)\,dy\,dx \\
&= \int_0^1\left[x^2y+\frac{1}{3}y^3+4y\Big|_{y=0}^{y=1-x}\right]dx \\
&= \int_0^1\left[x^2(1-x)+\frac{1}{3}(1-x)^3+4(1-x)\right]dx \\
&= \int_0^1 x^2-x^3+\frac{1}{3}(1-3x+3x^2-x^3)+4-4x\,dx \\
&= \int_0^1 \frac{13}{3}-5x+2x^2-\frac{4}{3}x^3\,dx \\
&= \frac{13}{3}x-\frac{5}{2}x^2+\frac{2}{3}x^3-\frac{1}{3}x^4\Big|_0^1 \\
&= \boxed{\frac{13}{6}}
\end{aligned}$$

8. Find the volume of the solid bounded by the cylinders $x^2+y^2=r^2$ and $y^2+z^2=r^2$.

By symmetry, the volume of the solid is 8 times V_1, which is the volume of the solid just in the first octant. The solid in the first octant is bounded by the xy-plane, $x=0$, $y=0$, $x=\sqrt{r^2-y^2}$ and the surface $z^2=r^2-y^2$ which in the first octant is $z=\sqrt{r^2-y^2}$. In other words, we have V_1 is the volume of the solid over region R pictured below under the surface $z=\sqrt{r^2-y^2}$.



$$\begin{aligned}
V_1 &= \int_0^r\int_0^{\sqrt{r^2-y^2}}\sqrt{r^2-y^2}\,dx\,dy \\
&= \int_0^r\left[\sqrt{r^2-y^2}\,x\Big|_{x=0}^{x=\sqrt{r^2-y^2}}\right]dy \\
&= \int_0^r (r^2-y^2)\,dy \\
&= r^2y-\frac{1}{3}y^3\Big|_0^r \\
&= r^3-\frac{1}{3}r^3 = \frac{2}{3}r^3 \\
V &= 8V_1 = \boxed{\frac{16}{3}r^3}
\end{aligned}$$

9. Evaluate $\iint_R e^{x^2+y^2}\,dA$ where $R = \{(x,y)\,|\,16 \le x^2+y^2 \le 25,\ x \ge 0,\ y \ge 0\}$.

$$R = \left\{(r,\theta) \mid r \in [4,5],\ \theta \in \left[0, \frac{\pi}{2}\right]\right\}$$

$$\begin{aligned}
\iint_R e^{x^2+y^2}\,dA &= \int_0^{\frac{\pi}{2}} \int_4^5 e^{r^2} r\,dr\,d\theta \\
&= \int_0^{\frac{\pi}{2}} \left[\frac{1}{2}e^{r^2}\Big|_{r=4}^{r=5}\right] d\theta \\
&= \int_0^{\frac{\pi}{2}} \left[\frac{e^{25}-e^{16}}{2}\right] d\theta \\
&= \boxed{\frac{(e^{25}-e^{16})\pi}{4}}
\end{aligned}$$

10. Use polar coordinates to find the volume of the solid bounded by the paraboloid $z = 10 - 3x^2 - 3y^2$ and the plane $z = 4$.

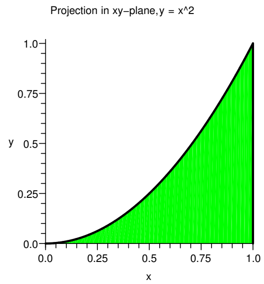
The paraboloid intersects the plane $z = 4$ when $4 = 10 - 3(x^2+y^2)$ or $x^2+y^2 = r^2 = 2$

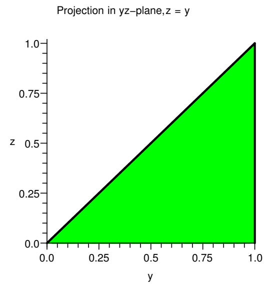
$$\begin{aligned}
V &= \iint_{x^2+y^2\le 2} [10 - 3(x^2+y^2) - 4]dA \\
&= \int_0^{2\pi} \int_0^{\sqrt{2}} (6-3r^2)r\,dr\,d\theta \\
&= \int_0^{2\pi} \int_0^{\sqrt{2}} 6r - 3r^3\,dr\,d\theta \\
&= \int_0^{2\pi} \left[3r^2 - \frac{3}{4}r^4\Big|_0^{\sqrt{2}}\right] d\theta \\
&= \int_0^{2\pi} 6 - \frac{3}{4}\cdot 4 - 0\,d\theta = \int_0^{2\pi} 3\,d\theta \\
&= \boxed{6\pi}
\end{aligned}$$

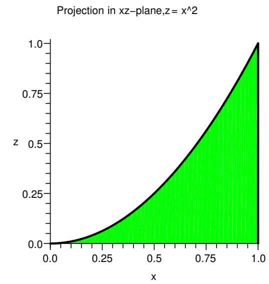
11. Evaluate $\int_0^1 \int_0^z \int_0^y ze^{-y^2}\,dx\,dy\,dz$.

$$\begin{aligned}
\int_0^1 \int_0^z \int_0^y ze^{-y^2}\,dx\,dy\,dz &= \int_0^1 \int_0^z \left[xze^{-y^2}\Big|_{x=0}^{x=y}\right] dy\,dz \\
&= \int_0^1 \int_0^z yze^{-y^2}\,dy\,dz \\
&= \int_0^1 \left[-\frac{1}{2}ze^{-y^2}\Big|_{y=0}^{y=z}\right] dz \\
&= \int_0^1 \left[-\frac{1}{2}ze^{-z^2} + \frac{1}{2}z\right] dz \\
&= \frac{1}{4}e^{-z^2} + \frac{1}{4}z^2\Big|_0^1 = \frac{1}{4}e^{-1} + \frac{1}{4} - \frac{1}{4} \\
&= \boxed{\frac{1}{4}e^{-1} = \frac{1}{4e}}
\end{aligned}$$

12. Rewrite the integral $\int_0^1 \int_0^{x^2} \int_0^y f(x,y,z)\,dz\,dy\,dx$ as an equivalent iterated integral in five other orders.







Above are the projections of the region E onto the xy, yz and xz-planes, respectively and are described below mathematically.

$$
\begin{aligned}
D_1 &= \{(x,y)\,|\,x \in [0,1], 0 \le y \le x^2\} = \{(x,y)\,|\,y \in [0,1], \sqrt{y} \le x \le 1\} \\
D_2 &= \{(y,z)\,|\,y \in [0,1], 0 \le z \le y\} = \{(y,z)\,|\,z \in [0,1], z \le y \le 1\} \\
D_3 &= \{(x,z)\,|\,x \in [0,1], 0 \le z \le x^2\} = \{(x,z)\,|\,z \in [0,1], \sqrt{z} \le x \le 1\} \\
\Longrightarrow E &= \{(x,y,z)\,|\,(x,y) \in D_1,\ 0 \le z \le y,\} \\
&= \{(x,y,z)\,|\,(y,z) \in D_2, \sqrt{y} \le x \le 1\} \\
&= \{(x,y,z)\,|\,(x,z) \in D_3, z \le y \le x^2\} \\
\Longrightarrow \int_0^1 \int_0^{x^2} \int_0^y f(x,y,z)\,dz\,dy\,dx &= \int_0^1 \int_{\sqrt{y}}^1 \int_0^y f(x,y,z)\,dz\,dx\,dy \\
&= \int_0^1 \int_0^y \int_{\sqrt{y}}^1 f(x,y,z)\,dx\,dz\,dy \\
&= \int_0^1 \int_z^1 \int_{\sqrt{y}}^1 f(x,y,z)\,dx\,dy\,dz \\
&= \int_0^1 \int_0^{x^2} \int_z^{x^2} f(x,y,z)\,dy\,dz\,dx \\
&= \int_0^1 \int_{\sqrt{z}}^1 \int_z^{x^2} f(x,y,z)\,dy\,dx\,dz
\end{aligned}
$$

13. Evaluate $\iiint_D \pi yz\cos\left(\frac{\pi}{2}x^5\right)dV$ where $D = \{(x,y,z)\,|\,1 \le x \le 2,\ 0 \le y \le x,\ x \le z \le 2x\}$.

$$\begin{aligned}
\iiint_D \pi yz\cos\left(\frac{\pi}{2}x^5\right)dV &= \int_1^2\int_0^x\int_x^{2x} \pi yz\cos\left(\frac{\pi}{2}x^5\right)\,dz\,dy\,dx \\
&= \int_1^2\int_0^x \left[\frac{\pi}{2}yz^2\cos\left(\frac{\pi}{2}x^5\right)\Big|_{z=x}^{z=2x}\right]dy\,dx \\
&= \int_1^2\int_0^x \left[\frac{\pi}{2}y4x^2\cos\left(\frac{\pi}{2}x^5\right) - \frac{\pi}{2}yx^2\cos\left(\frac{\pi}{2}x^5\right)\right]dy\,dx \\
&= \int_1^2\int_0^x \frac{3\pi}{2}yx^2\cos\left(\frac{\pi}{2}x^5\right)dy\,dx \\
&= \int_1^2 \left[\frac{3\pi}{4}y^2x^2\cos\left(\frac{\pi}{2}x^5\right)\Big|_{y=0}^{y=x}\right]dx \\
&= \int_1^2 \frac{3\pi}{4}x^4\cos\left(\frac{\pi}{2}x^5\right)dx = \frac{3}{10}\sin\left(\frac{\pi}{2}x^5\right)\Big|_1^2 \\
&= \frac{3}{10}\left[\sin(16\pi) - \sin\frac{\pi}{2}\right] = \boxed{-\frac{3}{10}}
\end{aligned}$$

14. Evaluate $\iiint_E x\,dV$ where E is the solid enclosed by the planes $z = 0$ and $z = x + y + 3$ and by the cylinders $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

$$E = \{(r,\theta,z)\,|\,r \in [2,3], \theta \in [0,2\pi], 0 \le z \le r\cos\theta + r\sin\theta + 3\}$$

$$\begin{aligned}
\iiint_E x\,dV &= \int_0^{2\pi}\int_2^3\int_0^{r\cos\theta + r\sin\theta + 3} (r\cos\theta)r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_2^3\int_0^{r\cos\theta + r\sin\theta + 3} r^2\cos\theta\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_2^3 \left[r^2\cos\theta z\Big|_{z=0}^{z=r\cos\theta + r\sin\theta + 3}\right]dr\,d\theta \\
&= \int_0^{2\pi}\int_2^3 r^3(\cos^2\theta + \cos\theta\sin\theta) + 3r^2\cos\theta\,dr\,d\theta \\
&= \int_0^{2\pi} \left[\frac{1}{4}r^4(\cos^2\theta + \cos\theta\sin\theta) + r^3\cos\theta\right]_{r=2}^{r=3} d\theta \\
&= \int_0^{2\pi} \left[\left(\frac{81}{4} - \frac{16}{4}\right)(\cos^2\theta + \cos\theta\sin\theta) + (27-8)\cos\theta\right] d\theta \\
&= \int_0^{2\pi} \left[\frac{65}{4}\left(\frac{1}{2}(1 + \cos 2\theta) + \frac{1}{2}\sin 2\theta\right) + 19\cos\theta\right] d\theta \\
&= \frac{65}{8}(\theta + \frac{1}{2}\sin 2\theta) - \frac{65}{16}\cos 2\theta + 19\sin\theta\Big|_0^{2\pi} \\
&= \frac{65}{8}(2\pi + 0) - \frac{65}{16} + 0 - \left(0 - \frac{65}{16} + 0\right) \\
&= \boxed{\frac{65\pi}{4}}
\end{aligned}$$

15. Evaluate $\iiint_E e^{\sqrt{x^2+y^2+z^2}} dV$, where E is enclosed by the sphere $x^2+y^2+z^2=9$ in the first octant.

$$E = \{(\rho,\theta,\phi)\,|\,\rho\in[0,3],\ \theta\in[0,\pi/2],\ \phi\in[0,\pi/2]\}$$

$$\iiint_E e^{\sqrt{x^2+y^2+z^2}}\,dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^3 e^{\sqrt{\rho^2}}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^3 \rho^2 e^\rho \sin\phi\,d\rho\,d\phi\,d\theta$$

by formula on pg 963, extended for triple integrals:

$$= \left[\int_0^{\pi/2} d\theta\right]\left[\int_0^{\pi/2}\sin\phi\,d\phi\right]\left[\int_0^3 \rho^2 e^\rho\,d\rho\right]$$

int by parts twice on third integral:

$$= [\theta]_0^{\pi/2}\ [-\cos\phi]_0^{\pi/2}\ \left[\rho^2e^\rho - 2\rho e^\rho + 2e^\rho\right]_0^3$$
$$= \left(\frac{\pi}{2}\right)(0-(-1))\left(9e^3-6e^3+2e^3-(0-0+2)\right)$$
$$= \boxed{\frac{\pi}{2}(5e^3-2)}$$

16. Consider the two surfaces $\rho = 3\csc\theta$ in spherical coordinates and $r=3$ in cylindrical coordinates. Are they the same surface, or different? Explain your answer.

Several ways to do this, this is one:

For the first surface, remember:

$$x = \rho\cos\theta\sin\phi$$
$$y = \rho\sin\theta\sin\phi$$
$$z = \rho\cos\phi$$

For the second surface, remember:

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

So if we let $\phi = 0$ in the first surface, we would get $x=0$, and $y=0$. For the second surface, when $x=0$, that would mean $\cos\theta = 0$ since $r=3$. Then on the second surface, by Pythagorean Identity, if $\cos\theta=0$, then $\sin\theta=\pm1$. Thus $y=\pm3$ when $x=0$. But on the first surface $x=0$ and $y=0$. Thus these cannot be the same surface.