GENERALISED INVERSES IN JÖRGENS ALGEBRAS OF BOUNDED LINEAR OPERATORS

By Lisa A. Oberbroeckling
Loyola College in Maryland, Baltimore, Maryland 21210, USA

[Received 10 March 2005. Read 12 July 2005. Published 14 February 2006.]

ABSTRACT

Let \( X \) be a Banach space and \( T \) be a bounded linear operator from \( X \) to itself \((T \in B(X))\). An operator \( S \in B(X) \) is a generalised inverse of \( T \) if \( TST = T \). In this paper we look at the Jörgens algebra, an algebra of operators on a dual system, and characterise when an operator in that algebra has a generalised inverse that is also in the algebra. This result is then applied to bounded inner product spaces and \(^*\)-algebras.

1. Introduction

Let \( B(X) \) denote the space of bounded linear operators from a complex Banach space \( X \) to itself. An operator \( T \in B(X) \) has a generalised inverse \( S \in B(X) \) if \( TST = T \). If \( X \) is finite-dimensional \((X = \mathbb{C}^n)\), every operator in \( B(X) = M_n(\mathbb{C}) \) has a generalised inverse. If not, \( T \) may or may not have a generalised inverse. Under the conditions where \( X \) is infinite-dimensional, the characterisation of when an operator \( T \in B(X) \) has a generalised inverse in \( B(X) \) and methods of the construction of a generalised inverse are well-known [2; 8].

In Section 2 we look at an important Banach algebra of bounded linear operators called the Jörgens algebra. This algebra is so named because K. Jörgens presented this algebra in [6] as a way to study integral operators. There Jörgens studied the spectral and Fredholm theory relative to a Jörgens algebra. This topic was also studied by B. Barnes in [1].

Let \((X, \| \cdot \|_X)\) and \((Y, \| \cdot \|_Y)\) be Banach spaces in normed duality. That is, suppose there is a nondegenerate bilinear form \( \langle \cdot, \cdot \rangle \) on \( X \times Y \) such that for some \( M > 0 \),
\[
|\langle x, y \rangle| \leq M \| x \|_X \| y \|_Y \text{ for all } x \in X \text{ and } y \in Y. \tag{1.1}
\]

Suppose \( T \in B(X) \) has an adjoint with respect to this bilinear form denoted by \( T^\dagger \); i.e., \( \langle Tx, y \rangle = \langle x, T^\dagger y \rangle \) for all \( x \in X \) and \( y \in Y \). Define the Jörgens algebra \( J_Y(X) = \mathcal{A} \) to be
\[
\mathcal{A} = \{ T \in B(X) : T^\dagger \text{ exists in } B(Y) \} \text{ with norm } \| T \| = \max\{ \| T \|_{op}, \| T^\dagger \|_{op} \}.
\]

With this defined norm, \( \mathcal{A} \) is a Banach algebra [6]. \( \mathcal{A} \) will denote the Jörgens algebra. Because the bilinear form is nondegenerate, an operator \( T \) in \( \mathcal{A} \) is uniquely

*E-mail: loberbroeckling@loyola.edu

Mathematical Proceedings of the Royal Irish Academy, 106A (1), 85–95 (2006) © Royal Irish Academy
determined by $T^\dagger$ and vice-versa. Note that a Jörgens algebra is a saturated algebra, or, more specifically, a $Y$-saturated algebra [5; 6, exercise 3.18].

In [6], Jörgens characterises when an operator $T \in A$ has a Fredholm inverse in $A$; see [6, section 5.8 and in particular theorem 5.16]. As Jörgens shows, in this case $T$ also has a generalised inverse in $A$. In Section 2 of this paper we study the more general question:

Under what conditions does an operator $T \in A$ have a generalised inverse $S \in A$?

The answer to this question is the main result of this paper.

In Section 3 we extend the main result in Section 2 to the *-algebra of bounded linear operators that have an adjoint with respect to an inner product on a Banach space.

2. The Jörgens Algebra

Inequality 1.1 in the previous section gives us continuity of the bilinear form for a fixed $y \in Y$ or a fixed $x \in X$. Thus we can identify $y \in Y$ with an element $\alpha y$ in the dual space of $X$ (denoted $X^*$) by $\alpha y(x) = \langle x, y \rangle$, and likewise we can identify $x \in X$ with an element $\beta x \in Y^*$. By nondegeneracy of the bilinear form, $Y$ is a total subspace of $X^*$ and $X$ is a total subspace of $Y^*$. Weak topologies, the $Y$-topology on $X$ and the $X$-topology on $Y$, are formed as in [3], and these topologies are locally convex.

Thus we have for nets $\{x_\delta\} \subseteq X$ and $\{y_\gamma\} \subseteq Y$ the following meaning of convergence in these topologies:

$x_\delta \xrightarrow{Y} x_o$ means $\langle x_\delta, y \rangle \longrightarrow \langle x_o, y \rangle \forall y \in Y$;

$y_\gamma \xrightarrow{X} y_o$ means $\langle x, y_\gamma \rangle \longrightarrow \langle x, y_o \rangle \forall x \in X$.

Clearly if $Y = X^*$ then the $Y$-topology is exactly the usual weak topology and the $X$-topology is the weak* topology.

Both the $X$-topology and $Y$-topology play an important role in studying generalised inverses in the Jörgens algebra. Using [3, theorem V.3.9], we prove the following result pertaining to the Jörgens algebra and the $X$- and $Y$-topologies.

**Theorem 2.1.** An operator $T \in B(X)$ is $Y$-continuous if and only if $T \in A$. Likewise for $S \in B(Y)$, $S$ is $X$-continuous if and only if $S = T^\dagger$ for some $T \in A$.

**Proof.** First suppose that $T \in A$ and let $\{x_\delta\}$ be any net in $X$ such that $x_\delta \xrightarrow{Y} x_o$ for some $x_o \in X$. We then have

$\langle T x_\delta, y \rangle = \langle x_\delta, T^\dagger y \rangle \longrightarrow \langle x_o, T^\dagger y \rangle = \langle T x_o, y \rangle$ for all $y \in Y$.

Thus $T x_\delta \xrightarrow{Y} T x_o$ so $T$ is $Y$-continuous.

Now suppose that $T$ is $Y$-continuous. Then for each net $\{x_\delta\} \subseteq X$ such that $x_\delta \xrightarrow{Y} x_o$ we have $T x_\delta \xrightarrow{Y} T x_o$. In other words, $\langle T x_\delta, y \rangle \longrightarrow \langle T x_o, y \rangle$ for each $y \in Y$. Thus the linear functionals on $X$ defined by $\alpha y(x) := \langle T x, y \rangle$ for each $y \in Y$
are continuous in the $\mathcal{J}$-topology. By [3, theorem V.3.9], for each $y \in Y$ there exists a corresponding unique $y' \in Y$ such that $\alpha_y(x) = \langle x, y' \rangle$ for each $x \in X$. Define $T' : Y \to Y$ by $T'y := y'$. Clearly $T'$ is well-defined and linear by nondegeneracy and linearity of $\langle \cdot, \cdot \rangle$. Also it is clear that

$$\langle Tx, y \rangle = \langle x, y' \rangle = \langle x, T'y \rangle$$

for each $x \in X$ and $y \in Y$.

To show $T' \in B(Y)$ it is enough to show that $T'$ is closed by the Closed Graph theorem. Let $\{y_n\}$ be a sequence in $Y$, $y_o$ and $y$ elements in $Y$ such that

$$\|y_n - y_o\| \to 0 \quad \text{and} \quad \|T'y_n - y\| \to 0 \quad \text{as} \quad n \to \infty.$$

Then for any $x \in X$,

$$|\langle x, T'y_o - y \rangle| = |\langle x, T'(y_o - y_n) \rangle + \langle x, T'y_n - y \rangle|$$

$$\leq M \|T\|_{op} \|x\| \|y_o - y_n\| + M \|x\| \|T'y_n - y\| \to 0.$$

Thus $|\langle x, T'y_o - y \rangle| = 0$ for all $x \in X$. By nondegeneracy of the form $T'y_o = y$ so $T'$ is a closed map and so is continuous. Therefore, $T \in A$ with $T^\dagger = T'$.

Similarly, the result for $S \in B(Y)$ can be shown. \(\blacksquare\)

For subspaces $A \subseteq X$ and $B \subseteq Y$ we have perp-spaces $A^\perp \subseteq Y$ and $^\perp B \subseteq X$ defined as

$$A^\perp = \{y \in Y \mid \langle x, y \rangle = 0 \quad \text{for all} \quad x \in A\} \quad \text{and} \quad ^\perp B = \{x \in X \mid \langle x, y \rangle = 0 \quad \text{for all} \quad y \in B\}.$$

It is not hard to show that $A^\perp$ is both norm and $X$-closed and $^\perp B$ is both norm and $\mathcal{J}$-closed.

**Lemma 1.** Let $M$ be a subspace of $X$ and $N$ a subspace of $Y$.

1. $^\perp (M^\perp)$ is the $\mathcal{J}$-closure of $M$ and $(^\perp N)^\perp$ is the $X$-closure of $N$; i.e. $^\perp (M^\perp) = \text{cl}_\mathcal{J} M$ and $(^\perp N)^\perp = \text{cl}_X N$;
2. $M = \text{cl}_\mathcal{J} M$ if and only if $^\perp (M^\perp) = M$ and similarly $N = \text{cl}_X N$ if and only if $^\perp (^\perp N)^\perp = N$;
3. for any $T \in A$, $N(T) = \text{cl}_\mathcal{J} N(T)$ and $N(T^\dagger) = \text{cl}_X N(T^\dagger)$; and
4. for any $T \in A$, $\mathcal{R}(T^\dagger) \subseteq N(T^\perp)$ and $\mathcal{R}(T) \subseteq ^\perp N(T^\perp)$.

The first two results are direct corollaries of the Hahn-Banach theorem while the third follows from the Hahn-Banach theorem and Theorem 2.1. The fourth result is clear.

For an operator $T \in A$, one can consider when

$$\mathcal{R}(T)^\perp = N(T^\dagger), \quad N(T)^\perp = \mathcal{R}(T^\dagger),$$

$$^\perp \mathcal{R}(T^\dagger) = N(T), \quad \text{and} \quad ^\perp N(T^\dagger) = \mathcal{R}(T).$$
Lemma 2. Let \( T \in A \).

(1) \( \mathcal{R}(T) = \mathcal{N}(T) \);  
(2) \( \mathcal{N}(T) = \mathcal{R}(T) \);  
(3) \( \mathcal{N}(T) = \text{cl}_Y \mathcal{R}(T) \); and  
(4) \( \mathcal{N}(T) = \mathcal{R}(T) \).

Proof. Clearly \( \mathcal{N}(T) \subseteq \mathcal{R}(T) \). Let \( y \in \mathcal{R}(T) \) be arbitrary. Then
\[
\langle x, T^*y \rangle = \langle Tx, y \rangle = 0 \quad \text{for all } x \in X.
\]
By nondegeneracy of the form, \( T^*y = 0 \) so \( y \in \mathcal{N}(T) \). By a similar argument, \( \mathcal{R}(T) \) is also \( \mathcal{N}(T) \). From these two equalities we get\( \mathcal{R}(T) \) is also \( \mathcal{N}(T) \). Thus \( \mathcal{R}(T) \) is also \( \mathcal{N}(T) \). From Lemma 1 we obtain the last two results of the lemma. \( \blacksquare \)

We now have the following useful lemma.

Lemma 3. The following are true for any projection \( P \in A \):

(1) \( \mathcal{N}(P) = \mathcal{R}(P) \);  
(2) \( \mathcal{R}(P) = \mathcal{N}(P) \);  
(3) \( \mathcal{R}(P) = \mathcal{N}(P) \); and  
(4) \( \mathcal{N}(P) = \mathcal{R}(P) \).

Thus \( \mathcal{R}(P) \) and \( \mathcal{N}(P) \) are both \( \mathcal{Y} \)-closed and \( \mathcal{R}(P) \) and \( \mathcal{N}(P) \) are both \( \mathcal{X} \)-closed.

Proof. To prove the first two notice that both \( P \) and \( I - P \) are in \( A \). From Lemma 1, both \( \mathcal{N}(P) \) and \( \mathcal{N}(I - P) = \mathcal{R}(P) \) are \( \mathcal{Y} \)-closed and thus Lemma 2 applies. The last two equalities use the same argument on \( P^* \) and \( I - P^* \). \( \blacksquare \)

We immediately have the following theorem.

Theorem 2.2. Let \( P \) be a projection in \( B(X) \). Then \( Y = \mathcal{R}(P) \oplus \mathcal{N}(P) \) if and only if \( P \in A \).

Proof. First assume that \( Y = \mathcal{R}(P) \oplus \mathcal{N}(P) \). Then for any \( x \in X \) and \( y \in Y \) we have unique representations
\[
x = x_1 + x_2, \quad x_1 \in \mathcal{R}(P), \quad x_2 \in \mathcal{N}(P) \quad \text{and}  
y = y_1 + y_2, \quad y_1 \in \mathcal{R}(P), \quad y_2 \in \mathcal{N}(P).  
\]
Note that \( \langle x_1, y_1 \rangle = 0 \) and \( \langle x_2, y_2 \rangle = 0 \). Since \( \mathcal{N}(P) \) and \( \mathcal{R}(P) \) are both norm-closed subspaces, we can define \( Q \in B(Y) \) to be the continuous projection onto \( \mathcal{N}(P) \) with nullspace \( \mathcal{R}(P) \) [8, theorem IV.12.2]. Then for any \( x \in X \) and \( y \in Y \) and the above representations,
\[
\langle x, Qy \rangle = \langle x_1, y_2 \rangle  
= \langle x_1, y_2 \rangle + \langle x_2, y_2 \rangle
\]
Let there exist projections $R$ and $Q$. By the above lemma, $R$ and $Q$ are continuous projections.

Theorem 2.3. The main result of this paper is the following.

Result characterises the existence of generalised inverses in the Jörgens algebra and bounded, one-to-one operator onto $T$. By Lemma 3, $\langle P^*x, y \rangle = 1$. Let $S(T) = \{ x \mid P^*x = 0 \}$. Then $S(T)$ is a projection onto $S(T)$. Clearly, by construction, $P^*S(T) = S(T)$. However, by Lemma 3, $N(P^*) = R(P^*) = N(T)^\perp$. Thus $Y = R(P^*) \oplus N(P^*)$.

If an operator $T \in A$ is invertible in $A$, it is clear that $(T^{-1})^\perp$ must equal $(T^\perp)^{-1}$. Thus $T^\perp$ must also be invertible in $B(Y)$. To consider when an arbitrary operator in $A$ has a generalised inverse in $A$ we must consider the different topologies on $X$ and $Y$ and how the nullspaces and ranges of $T$ and $T^\perp$ are related. The following result characterises the existence of generalised inverses in the Jörgens algebra and is the main result of this paper.

Theorem 2.3. Let $T \in A$. $T$ has a generalised inverse $S \in A$ if and only if

1. There exist projections $P$ and $Q$ in $A$ such that
   
   $\mathcal{R}(P) = N(T)$, $\mathcal{R}(Q) = R(T)$; and
   
2. $\mathcal{R}(T^\perp) = N(T)^\perp$.

Proof. First assume that there exists a generalised inverse $S$ of $T$ such that $S \in A$. Then $S^\perp$ is a generalised inverse of $T^\perp$. By [8, theorem IV.12.9], there exist continuous projections $P = I - ST$ and $Q = TS$ such that $\mathcal{R}(P) = N(T)$ and $\mathcal{R}(Q) = R(T)$. Clearly, by construction, $P$ and $Q$ are in $A$ with $P^\perp = I - T^\perp S^\perp$ and $Q^\perp = S^\perp T^\perp$. By that same theorem, $T^\perp S^\perp$ is a projection onto $\mathcal{R}(T^\perp)$; thus $N(P^\perp) = \mathcal{R}(T^\perp)$. However, by Lemma 3, $N(P^\perp) = \mathcal{R}(P^\perp) = N(T)^\perp$. Therefore, $\mathcal{R}(T^\perp) = N(T)^\perp$.

Conversely, suppose $\mathcal{R}(T^\perp) = N(T)^\perp$ and let $P$ and $Q$ be the projections as in the hypothesis (1). Let $L = N(P)$ and $M = N(Q)$. Then

$X = N(T) \oplus L = \mathcal{R}(T) \oplus M$

and from Theorem 2.2,

$Y = N(T)^\perp \oplus L^\perp = \mathcal{R}(T)^\perp \oplus M^\perp$.

By Lemma 3,

$N(P^\perp) = \mathcal{R}(P)^\perp = N(T)^\perp = \mathcal{R}(T^\perp)$, $\mathcal{R}(P^\perp) = L^\perp$,

$N(Q^\perp) = \mathcal{R}(Q)^\perp = R(T)^\perp = N(T^\perp)$, and $\mathcal{R}(Q^\perp) = M^\perp$.

Define the map $T_1 : L \to \mathcal{R}(T)$ by $T_1 x = Tx$ for all $x \in L$. Clearly $T_1$ is a linear, bounded, one-to-one operator onto $\mathcal{R}(T)$. Since $\mathcal{R}(T)$ is norm-closed, it is a Banach
space. Thus by the Open Mapping theorem, $T_1^{-1} : \mathcal{R}(T) \rightarrow L$ exists as a bounded linear operator. Define $S : X \rightarrow X$ to be $T_1^{-1}Q$. Clearly $S \in B(X)$ and $STx = x$ for all $x \in L$. Let $x \in X$ be arbitrary. Since $x$ can be expressed uniquely as $x = x_1 + x_2$ with $x_1 \in \mathcal{N}(T)$ and $x_2 \in L$,

$$TSTx = TST(x_1 + x_2) = TSTx_2 = Tx_2 = T(x_1 + x_2) = Tx.$$  

Thus $S$ is a generalised inverse for $T$.

Define $T_2 : M^\perp \rightarrow \mathcal{R}(T^\dagger)$ by $T_2y = T^\dagger y$. Clearly $T_2$ is a bounded linear operator. It is one-to-one and onto $\mathcal{R}(T^\dagger)$ since $\mathcal{R}(T^\dagger) = \mathcal{N}(T^\dagger)$. Since $\mathcal{R}(T^\dagger)$ is $X$-closed, it is norm-closed so $T_2^\perp : \mathcal{R}(T^\dagger) \rightarrow M^\perp$ exists as a bounded linear operator by the Open Mapping theorem. Define $S_2 \in B(Y)$ by $S_2 = T_2^{-1}(I - P^\dagger)$. For any $y \in Y$ we have the unique representation $y = y_1 + y_2$ with $y_1 \in \mathcal{N}(T^\dagger) = \mathcal{R}(T^\dagger)$ and $y_2 \in M^\perp$. Note that

$$T^\dagger y_2 = T_2y_2 \quad \text{and} \quad S_2T^\dagger y_2 = T_2^{-1}T^\dagger y_2 = T_2^{-1}T_2y_2 = y_2.$$  

Then we have for any $y \in Y$ with $y = y_1 + y_2$ as above,

$$T^\dagger S_2T^\dagger y = T^\dagger S_2T^\dagger(y_1 + y_2) = T^\dagger S_2T^\dagger y_2 = T^\dagger y_2 = T^\dagger(y_1 + y_2) = T^\dagger y.$$  

Thus $S_2$ is a generalised inverse of $T^\dagger$. For any $x \in X$ and $y \in Y$ we have

$$x = Tx_1 + x_2, x_1 \in L, \quad x_2 \in M \quad \text{and} \quad y = T^\dagger y_1 + y_2, y_1 \in M^\perp, \quad y_2 \in L^\perp.$$  

By Lemma 3, $\mathcal{N}(I - P^\dagger) = \mathcal{R}(P^\dagger) = L^\perp$ so $S_2y_2 = T_2^{-1}(I - P^\dagger)y_2 = 0$. Thus

$$S_2y = S_2T^\dagger y_1 + S_2y_2 = S_2T^\dagger y_1 = y_1, \quad \text{since} \quad y_1 \in M^\perp$$  

and

$$\langle Sx, y \rangle = \langle STx_1, y \rangle + \langle Sx_2, y \rangle = \langle x_1, T^\dagger y_1 \rangle + \langle Sx_2, y \rangle = \langle x_1, T^\dagger y_1 \rangle + \langle Sx_2, y \rangle = \langle x_1, T^\dagger y_1 \rangle = \langle x_1 + x_2, y_1 \rangle = \langle x, y \rangle.$$  

$$\langle x, S_2y \rangle.$$
Therefore $S$ is a generalised inverse of $T$ in $A$ with $S^\dagger = S_2$. 

**Remark 1.** With the above construction of $S = T_1^{-1}Q$, $STS = S$. This is because $TT_1^{-1}x = x$ for all $x \in R(T)$ and $QT = T$, since $R(Q) = R(T)$, so we have

$$STS = T_1^{-1}QTT_1^{-1}Q = T_1^{-1}TT_1^{-1}Q = T_1^{-1}Q = S.$$ 

From the theorem and Lemma 3 we obtain the following corollary.

**Corollary 1.** Let $T \in A$ such that $T$ has a generalised inverse $S \in A$. Then $R(T) = ^\perp N(T^\dagger)$ and $R(T^\dagger) = N(T)^\perp$.

3. Banach Spaces with Bounded Inner Product

Let $X$ be a Banach space with a bounded inner product $(\cdot, \cdot)$. For $T \in B(X)$, define $T^*$ to be the adjoint of $T$ with respect to the inner product. That is,

$$(Tx, y) = (x, T^*y) \text{ for all } x, y \in X.$$ 

Define the algebra $B = \{ T \in B(X) \mid \exists T^* \in B(X) \}$. This is equivalent to the algebra of all bounded linear operators on $X$ that have bounded extensions to the Hilbert space completion of $X$ [7]. Define a norm on the elements of $B$ similar to the Jörgens algebra; that is, for $T \in B$,

$$\| T \| = \max\{ \| T \|_{op}, \| T^* \|_{op} \}.$$ 

This makes $B$ a Banach $*$-algebra and so Moore-Penrose inverses can be discussed. If $B$ is a $*$-algebra, $b \in B$ is a Moore-Penrose inverse of $a \in B$ if

$$aba = a, \quad bab = b, \quad (ba)^* = ba \quad \text{and} \quad (ab)^* = ab.$$ 

Throughout the rest of this section, $B$ will denote the $*$-algebra above with the inner product space $X$ and $T^*$ will denote the adjoint of $T$ in this algebra. As in the Jörgens algebra case we can define the space $M^\perp \subseteq X$ for a subspace $M$ of $X$. For a fixed $x_0 \in X$ define $\alpha_{x_0}(x) := (x, x_0)$. This is clearly a linear functional and by continuity of the inner product, $\alpha_{x_0} \in X^*$. Thus we have a weak $X$-topology on $X$ as defined in [3] and the Jörgens algebra case. All of the results about the $M^\perp$ spaces and the $X$-topology in the Jörgens algebra case apply. In particular we have the following results.

**Lemma 4.** The following are true for any projection $P \in B$.

1. $\mathcal{N}(P) = R(P^\dagger)^\perp$;
2. $\mathcal{R}(P) = N(P^\dagger)^\perp$;
3. $\mathcal{R}(P^\dagger) = \mathcal{N}(P)^\perp$; and
Let there exist projections $P, Q$ and $R$ with $P$, $Q$ and $R$ have Moore-Penrose inverses and $\mathcal{N}(P)$, $\mathcal{N}(P^*)$ and $\mathcal{N}(P^*)$ are all $\mathcal{X}$-closed.

**Theorem 3.1.** Let $P$ be a projection in $B(X)$. $P \in B$ if and only if $X = \mathcal{R}(P) \ominus \mathcal{N}(P^*)$.

**Theorem 3.2.** Let $T \in B$. $T$ has a generalised inverse in $B$ if and only if

1. There exist projections $P$ and $Q$ in $B$ with $\mathcal{R}(P) = \mathcal{N}(T)$ and $\mathcal{R}(Q) = \mathcal{R}(T)$; and
2. $\mathcal{R}(T^*) = \mathcal{N}(T^*)$.

The proofs of these theorems are the same as the Jörgens algebra case since the only difference is that there is a sesquilinear form rather than a bilinear form. We immediately have the following result.

**Theorem 3.3.** Let $T \in B$. $T$ has a Moore-Penrose inverse in $B$ if and only if

1. $X = \mathcal{N}(T) \oplus \mathcal{N}(T^*) = \mathcal{R}(T) \oplus \mathcal{R}(T^*)$; and
2. $\mathcal{R}(T) = \mathcal{N}(T^*)$ and $\mathcal{R}(T^*) = \mathcal{N}(T)$.

**Proof.** First assume that $T$ has a Moore-Penrose inverse $S \in B$. By definition, $S$ is a generalised inverse of $T$ in $B$ and there exist self-adjoint projections $P = I - ST$ and $Q = TS$ in $B$ such that

$$\mathcal{R}(P) = \mathcal{N}(T) \quad \text{and} \quad \mathcal{R}(Q) = \mathcal{R}(T).$$

From Lemma 4,

$$\mathcal{N}(P) = \mathcal{N}(P^*) = \mathcal{R}(P) \ominus \mathcal{N}(T) \quad \text{and} \quad \mathcal{N}(Q) = \mathcal{N}(Q^*) = \mathcal{R}(Q) \ominus \mathcal{N}(T).$$

Thus $X = \mathcal{N}(T) \oplus \mathcal{N}(T^*) = \mathcal{R}(T) \ominus \mathcal{R}(T^*)$. By Theorem 3.2, $\mathcal{R}(T^*) = \mathcal{N}(T^*)$. By Lemma 4, $\mathcal{R}(Q) = \mathcal{R}(T)$ is $\mathcal{X}$-closed; thus $\mathcal{R}(T) = \mathcal{N}(T^*)$.

Now assume the converse. Let $P$ be the projection onto $\mathcal{N}(T)$ along $\mathcal{N}(T^*)$ and $Q$ be the projection onto $\mathcal{R}(T)$ along $\mathcal{R}(T^*)$. Clearly, by Theorem 3.1 both $P$ and $Q$ are in $B$, and from Lemma 4 we have

$$\mathcal{R}(P) = \mathcal{N}(P) \ominus \mathcal{N}(T), \mathcal{N}(P^*) = \mathcal{R}(P) \ominus \mathcal{N}(T),$$
$$\mathcal{N}(Q) = \mathcal{N}(Q^*) = \mathcal{R}(Q) \ominus \mathcal{N}(T^*).$$

Thus $P^* = P$ and $Q^* = Q$. By Theorem 3.2, $T$ has a generalised inverse $S \in B$ such that $P = I - ST$, $Q = TS$ and $STS = S$. But $I - (ST)^* = P^* = P = I - ST$; thus $ST = (ST)^*$. Also $(TS)^* = Q^* = Q = TS$. Thus $S \in B$ is a Moore-Penrose inverse of $T$. ■
As in the Jörgens algebra case, an operator $T$ is invertible in $B$ if and only if $T$ and $T^*$ are invertible in $B(X)$ [1, theorem 2.5]. Also, we say $T$ is Fredholm with respect to $B$, or $T \in \Phi_B$, when $T$ is invertible modulo finite rank operators in $B$; i.e. there exists an operator $S \in B$ and finite rank operators $F, G \in B$ such that $ST = I + F$ and $TS = I + G$. This definition was discussed in [1] and shown to be a natural definition. Also, $B$ can be shown to be a saturated algebra and these ideas of Fredholm operators and finite rank operators are equivalent [5, theorem 25.1]. Let $\nu(T)$ denote the Fredholm index of $T$, and $\Phi_B^0$ be the set of operators $T \in \Phi_B$ for which $\nu(T) = 0$. Also, $T \in \Phi_B$ if and only if $T$ and $T^* \in \Phi(X)$ and $\nu(T) = \nu(T^*) = 0$ [1, theorem 2.5]. The analogous condition [6] offers the natural definition of $\Phi_A$.

Elements of C*-algebras that have generalised inverses also have Moore-Penrose inverses [4, theorem 6]. The proof of this result uses the symmetric property that for any element $x$ of a C*-algebra, $I + x^*x$ is invertible. In $B$ we do not necessarily have symmetry so we first need some preliminaries.

**Lemma 5.** Let $P$ be a projection in $B$. Then $\mathcal{N}(P) = \mathcal{N}(PP^*P)$ and $\mathcal{N}(P^*) = \mathcal{N}(P^*PP^*)$.

**Proof.** We only need to prove the first equality. Clearly $\mathcal{N}(P) \subseteq \mathcal{N}(PP^*P)$. To prove the reverse inclusion we use the fact that for any subspace $M \subseteq X$, $M \cap M^\perp = \{0\}$. Let $PP^*Px = 0$. By Lemma 4, $P^*Px \in \mathcal{N}(P) \cap \mathcal{N}(P^*) = \mathcal{N}(P) \cap \mathcal{N}(P)^\perp = \{0\}$ and therefore $Px \in \mathcal{N}(P) \cap \mathcal{R}(P) = \mathcal{R}(P)^\perp \cap \mathcal{R}(P) = \{0\}$. Thus $x \in \mathcal{N}(P)$ and we have equality. \hfill \blacksquare

Note that the above lemma is true for any $T \in B$ such that $\mathcal{R}(T^*) = \mathcal{N}(T)^\perp$ and $\mathcal{R}(T) = \mathcal{N}(T^*)^\perp$ (the other two equalities from the analogue of Lemma 2 are true for any $T \in B$).

**Lemma 6.** Let $P$ be a projection in $B$ and $U = I + (P - P^*)^*(P - P^*) = I - (P - P^*)^2$. Then $U$ is injective.

**Proof.** Suppose $Ux = 0$ for some $x \in X$. By definition of $U$,

\[ x = Px + P^*x - PP^*x - P^*Px. \]

By multiplying the equation by $P$ and $P^*$ separately, we get both $PP^*Px = 0$ and $P^*PP^*x = 0$. By Lemma 5 $x \in \mathcal{N}(PP^*P) \cap \mathcal{N}(P^*PP^*) = \mathcal{N}(P) \cap \mathcal{N}(P^*)$. Consequently $\mathcal{N}(U) \subseteq \mathcal{N}(P) \cap \mathcal{N}(P^*)$.

Clearly $U \in B$ and $U^* = U$. So we have $\mathcal{R}(U) = \mathcal{R}(U^*) \subseteq \mathcal{N}(U)^\perp$ and $\mathcal{N}(U) = \mathcal{R}(U)^\perp$. Let $y \in \mathcal{N}(U)$. Then for all $x \in X$,

\[ 0 = \langle x, Uy \rangle = \langle Ux, y \rangle = \langle x - Px - P^*x + PP^*x + P^*Px, y \rangle = \langle x, y \rangle - \langle Px, y \rangle - \langle P^*x, y \rangle + \langle PP^*x, y \rangle + \langle P^*Px, y \rangle \]
Let \( \Phi \) be the set of all projections. Then \( \Phi \) is also a \(*\)-algebra. In fact, the proof works in any \(*\)-algebra where \( T \) has a Moore-Penrose inverse and the proof then follows. It should be noted that the Moore-Penrose inverse \( \hat{S} \) of \( T \) is constructed in the theorem as follows:

\[
\hat{S} = P^* PU^{-1} SQQ^* V^{-1}
\]

where \( S \) is the generalised inverse of \( T \) in \( \mathcal{B} \), \( P = ST \), \( Q = TS \), \( U = I + (P - P^*)^* (P - P^*) \) and \( V = I + (Q - Q^*)^* (Q - Q^*) \) are all in \( \Phi \). Clearly \( U \) and \( V \) are in \( \mathcal{B} \) and both self-adjoint. By the above lemma both \( U \) and \( V \) are injective so \( U \) and \( V \) are both invertible in \( B(X) \). However, \( U = U^* \) and \( V = V^* \), thus \( U \) and \( V \) are both invertible in \( \mathcal{B} \).

Now we apply the proof of [4, theorem 6]. By nondegeneracy of the inner product, \( \mathfrak{N}(P) \cap \mathfrak{N}(P^*) = 0 \) for all \( x \in X \). By nondegeneracy of the inner product, \( y = 0 \) and so \( U \) is injective.

**Theorem 3.4.** Let \( T \in \mathcal{B} \) such that \( T \) has a generalised inverse \( S \in \mathcal{B} \). Let \( P = ST \) and \( Q = TS \). If \( U = I - (P - P^*)^2 \) and \( V = I - (Q - Q^*)^2 \) are both surjective then \( T \) has a Moore-Penrose inverse \( \hat{S} \) in \( \mathcal{B} \) defined by \( \hat{S} = P^* PU^{-1} SQQ^* V^{-1} \).

**Proof.** Let \( S \in \mathcal{B} \) be the generalised inverse of \( T \in \mathcal{B} \) and let \( P = ST \) and \( Q = TS \).

Let \( U = I + (P - P^*)^* (P - P^*) = I - (P - P^*)^2 \) and \( V = I + (Q - Q^*)^* (Q - Q^*) = I - (Q - Q^*)^2 \). Clearly \( U \) and \( V \) are in \( \mathcal{B} \) and both self-adjoint. By the above lemma both \( U \) and \( V \) are injective so \( U \) and \( V \) are both invertible in \( B(X) \). However, \( U = U^* \) and \( V = V^* \), thus \( U \) and \( V \) are both invertible in \( \mathcal{B} \).

**Corollary 2.** Let \( T \in \mathcal{B} \) such that \( T \in \Phi \). Then \( T \) has a Moore-Penrose inverse in \( \mathcal{B} \).

**Proof.** Recall that since \( T \in \Phi \), \( T \in \Phi(X) \), \( T^* \in \Phi(X) \), and \( \iota(T) + \iota(T^*) = 0 \) [1, theorem 2.5]. Thus \( T \) has a generalised inverse \( S \in \Phi \) [6, theorem 5.16]. Clearly \( T^* \) and \( S^* \) are both in \( \Phi \). The projections \( P = ST \), \( Q = TS \), \( P^* \) and \( Q^* \) are all in \( \Phi \) with \( \mathfrak{N}(P) = \mathfrak{N}(T) \) and \( \mathfrak{R}(Q) = \mathfrak{R}(T) \).

Let \( U = I + (P - P^*)^* (P - P^*) = I - (P - P^*)^2 \) and \( V = I + (Q - Q^*)^* (Q - Q^*) = I - (Q - Q^*)^2 \). As above, \( U \) and \( V \) are in \( \mathcal{B} \) and both self-adjoint. Clearly \( PP^* \in \Phi \).

Note that \( I - P \) is of finite rank since \( \mathfrak{R}(I - P) = \mathfrak{N}(P) = \mathfrak{N}(T) \). The operator \( P^* P - (I - P)P^* \in \Phi \) since \( (I - P)P^* \) is of finite rank [8, theorem IV.13.4]. Using the same theorem shows that \( U = I - P + (P^* P - (I - P)P^*) \in \Phi \). A similar argument on \( Q \) shows that \( V \in \Phi \).

By Lemma 6 both \( U \) and \( V \) are injective and since both are of index zero the operators are also surjective. Thus, we apply the previous theorem to get the Moore-Penrose inverse \( \hat{S} \) of \( T \) defined by

\[
\hat{S} = P^* PU^{-1} SQQ^* V^{-1} \in \mathcal{B}.
\]
Acknowledgement

A special thank you to my PhD advisor, Dr Bruce A. Barnes of the University of Oregon.

References