GENERALISED INVERSES IN JÖRGENS ALGEBRAS OF BOUNDED LINEAR OPERATORS

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Abstract

Let X be a Banach space and T be a bounded linear operator from X to itself ($T \in B(X)$). An operator $S \in B(X)$ is a generalised inverse of T if $TST = T$. In this paper we look at the Jörgens algebra, an algebra of operators on a dual system, and characterise when an operator in that algebra has a generalised inverse that is also in the algebra. This result is then applied to bounded inner product spaces and *-algebras.

1. Introduction

Let $B(X)$ denote the space of bounded linear operators from a complex Banach space X to itself. An operator $T \in B(X)$ has a generalised inverse $S \in B(X)$ if $TST = T$. If X is finite-dimensional ($X = \mathbb{C}^n$), every operator in $B(X) = M_n(\mathbb{C})$ has a generalised inverse. If not, T may or may not have a generalised inverse. Under the conditions where X is infinite-dimensional, the characterisation of when an operator $T \in B(X)$ has a generalised inverse in $B(X)$ and methods of the construction of a generalised inverse are well-known [2; 8].

In Section 2 we look at an important Banach algebra of bounded linear operators called the Jörgens algebra. This algebra is so named because K. Jörgens presented this algebra in [6] as a way to study integral operators. There Jörgens studied the spectral and Fredholm theory relative to a Jörgens algebra. This topic was also studied by B. Barnes in [1].

Let $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$ be Banach spaces in normed duality. That is, suppose there is a nondegenerate bilinear form $\langle \cdot, \cdot \rangle$ on $X \times Y$ such that for some $M > 0$,

$$|\langle x, y \rangle| \leq M \ \| x \|_X \| y \|_Y \ \text{ for all } x \in X \text{ and } y \in Y. \tag{1.1}$$

Suppose $T \in B(X)$ has an adjoint with respect to this bilinear form denoted by $T^\dagger$; i.e., $\langle Tx, y \rangle = \langle x, T^\dagger y \rangle$ for all $x \in X$ and $y \in Y$. Define the *Jörgens algebra* $J_Y(X) = \mathcal{A}$ to be

$$\mathcal{A} = \{T \in B(X) : T^\dagger \text{ exists in } B(Y)\} \text{ with norm } \| T \| = \max\{\| T \|_{op}, \| T^\dagger \|_{op}\}.$$

With this defined norm, $\mathcal{A}$ is a Banach algebra [6]. $\mathcal{A}$ will denote the Jörgens algebra. Because the bilinear form is nondegenerate, an operator T in $\mathcal{A}$ is uniquely

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determined by $T^\dagger$ and vice-versa. Note that a Jörgens algebra is a saturated algebra, or, more specifically, a Y-saturated algebra [5; 6, exercise 3.18].

In [6], Jörgens characterises when an operator $T \in \mathcal{A}$ has a Fredholm inverse in $\mathcal{A}$; see [6, section 5.8 and in particular theorem 5.16]. As Jörgens shows, in this case T also has a generalised inverse in $\mathcal{A}$. In Section 2 of this paper we study the more general question:

Under what conditions does an operator $T \in \mathcal{A}$ have a generalised inverse $S \in \mathcal{A}$?

The answer to this question is the main result of this paper.

In Section 3 we extend the main result in Section 2 to the *-algebra of bounded linear operators that have an adjoint with respect to an inner product on a Banach space.

2. The Jörgens Algebra

Inequality 1.1 in the previous section gives us continuity of the bilinear form for a fixed $y \in Y$ or a fixed $x \in X$. Thus we can identify $y \in Y$ with an element α_y in the dual space of X (denoted X^*) by $\alpha_y(x) = \langle x, y\rangle$, and likewise we can identify $x \in X$ with an element $\beta_x \in Y^*$. By nondegeneracy of the bilinear form, Y is a total subspace of X^* and X is a total subspace of Y^*. Weak topologies, the $\mathcal{Y}$-topology on X and the $\mathcal{X}$-topology on Y, are formed as in [3], and these topologies are locally convex. Thus we have for nets $\{x_\delta\} \subseteq X$ and $\{y_\gamma\} \subseteq Y$ the following meaning of convergence in these topologies:

$$x_\delta \xrightarrow{\mathcal{Y}} x_o \text{ means } \langle x_\delta, y\rangle \longrightarrow \langle x_o, y\rangle \,\forall\, y \in Y;$$

$$y_\gamma \xrightarrow{\mathcal{X}} y_o \text{ means } \langle x, y_\gamma\rangle \longrightarrow \langle x, y_o\rangle \,\forall\, x \in X.$$

Clearly if $Y = X^*$ then the $\mathcal{Y}$-topology is exactly the usual weak topology and the $\mathcal{X}$-topology is the weak*-topology.

Both the $\mathcal{X}$-topology and $\mathcal{Y}$-topology play an important role in studying generalised inverses in the Jörgens algebra. Using [3, theorem V.3.9], we prove the following result pertaining to the Jörgens algebra and the $\mathcal{X}$- and $\mathcal{Y}$-topologies.

Theorem 2.1. *An operator $T \in B(X)$ is $\mathcal{Y}$-continuous if and only if $T \in \mathcal{A}$. Likewise for $S \in B(Y)$, S is $\mathcal{X}$-continuous if and only if $S = T^\dagger$ for some $T \in \mathcal{A}$.*

Proof. First suppose that $T \in \mathcal{A}$ and let $\{x_\delta\}$ be any net in X such that $x_\delta \xrightarrow{\mathcal{Y}} x_o$ for some $x_o \in X$. We then have

$$\langle Tx_\delta, y\rangle = \langle x_\delta, T^\dagger y\rangle \to \langle x_o, T^\dagger y\rangle = \langle Tx_o, y\rangle \text{ for all } y \in Y.$$

Thus $Tx_\delta \xrightarrow{\mathcal{Y}} Tx_o$ so T is $\mathcal{Y}$-continuous.

Now suppose that T is $\mathcal{Y}$-continuous. Then for each net $\{x_\delta\} \subseteq X$ such that $x_\delta \xrightarrow{\mathcal{Y}} x_o$ we have $Tx_\delta \xrightarrow{\mathcal{Y}} Tx_o$. In other words, $\langle Tx_\delta, y\rangle \to \langle Tx_o, y\rangle$ for each $y \in Y$. Thus the linear functionals on X defined by $\alpha_y(x) := \langle Tx, y\rangle$ for each $y \in Y$

are continuous in the $\mathcal{Y}$-topology. By [3, theorem V.3.9], for each $y \in Y$ there exists a corresponding unique $y' \in Y$ such that $\alpha_y(x) = \langle x, y' \rangle$ for each $x \in X$. Define $T' : Y \to Y$ by $T'y := y'$. Clearly T' is well-defined and linear by nondegeneracy and linearity of $\langle \cdot, \cdot \rangle$. Also it is clear that

$$\langle Tx, y \rangle = \langle x, y' \rangle = \langle x, T'y \rangle \text{ for each } x \in X \text{ and } y \in Y.$$

To show $T' \in B(Y)$ it is enough to show that T' is closed by the Closed Graph theorem. Let $\{y_n\}$ be a sequence in Y, y_o and y elements in Y such that

$$\| y_n - y_o \| \to 0 \quad \text{and} \quad \| T'y_n - y \| \to 0 \text{ as } n \to \infty.$$

Then for any $x \in X$,

$$\begin{aligned} |\langle x, T'y_o - y \rangle| &= |\langle x, T'(y_o - y_n) \rangle + \langle x, T'y_n - y \rangle| \\ &= |\langle Tx, y_o - y_n \rangle + \langle x, T'y_n - y \rangle| \\ &\leq M \ \| T \|_{op} \| x \| \| y_o - y_n \| + M \ \| x \| \| T'y_n - y \| \to 0. \end{aligned}$$

Thus $|\langle x, T'y_o - y \rangle| = 0$ for all $x \in X$. By nondegeneracy of the form $T'y_o = y$ so T' is a closed map and so is continuous. Therefore, $T \in \mathcal{A}$ with $T^\dagger = T'$.

Similarly, the result for $S \in B(Y)$ can be shown. ■

For subspaces $A \subseteq X$ and $B \subseteq Y$ we have perp-spaces $A^\perp \subseteq Y$ and ${}^\perp B \subseteq X$ defined as

$$\begin{aligned} A^\perp &= \{ y \in Y \,|\, \langle x, y \rangle = 0 \text{ for all } x \in A \} \quad \text{and} \\ {}^\perp B &= \{ x \in X \,|\, \langle x, y \rangle = 0 \text{ for all } y \in B \}. \end{aligned}$$

It is not hard to show that $A^\perp$ is both norm and $\mathcal{X}$-closed and ${}^\perp B$ is both norm and $\mathcal{Y}$-closed.

Lemma 1. *Let M be a subspace of X and N a subspace of Y.*

(1) *${}^\perp(M^\perp)$ is the $\mathcal{Y}$-closure of M and $({}^\perp N)^\perp$ is the $\mathcal{X}$-closure of N; i.e. ${}^\perp(M^\perp) = \mathrm{cl}_{\mathcal{Y}} M$ and $({}^\perp N)^\perp = \mathrm{cl}_{\mathcal{X}} N$;*

(2) *$M = \mathrm{cl}_{\mathcal{Y}} M$ if and only if ${}^\perp(M^\perp) = M$ and similarly $N = \mathrm{cl}_{\mathcal{X}} N$ if and only if $({}^\perp N)^\perp = N$;*

(3) *for any $T \in \mathcal{A}$, $\mathcal{N}(T) = \mathrm{cl}_{\mathcal{Y}} \mathcal{N}(T)$ and $\mathcal{N}(T^\dagger) = \mathrm{cl}_{\mathcal{X}} \mathcal{N}(T^\dagger)$; and*

(4) *for any $T \in \mathcal{A}$, $\mathcal{R}(T^\dagger) \subseteq \mathcal{N}(T)^\perp$ and $\mathcal{R}(T) \subseteq {}^\perp\mathcal{N}(T^\dagger)$.*

The first two results are direct corollaries of the Hahn-Banach theorem while the third follows from the Hahn-Banach theorem and Theorem 2.1. The fourth result is clear.

For an operator $T \in \mathcal{A}$, one can consider when

$$\begin{gathered} \mathcal{R}(T)^\perp = \mathcal{N}(T^\dagger), \qquad \mathcal{N}(T)^\perp = \mathcal{R}(T^\dagger), \\ {}^\perp\mathcal{R}(T^\dagger) = \mathcal{N}(T), \text{ and } {}^\perp\mathcal{N}(T^\dagger) = \mathcal{R}(T). \end{gathered}$$

Lemma 2. *Let $T \in \mathcal{A}$.*
(1) $\mathcal{R}(T)^\perp = \mathcal{N}(T^\dagger)$*;*
(2) ${}^\perp\mathcal{R}(T^\dagger) = \mathcal{N}(T)$*;*
(3) ${}^\perp\mathcal{N}(T^\dagger) = \mathrm{cl}_\mathcal{Y}\mathcal{R}(T)$*; and*
(4) $\mathcal{N}(T)^\perp = \mathrm{cl}_\mathcal{X}\mathcal{R}(T^\dagger)$*.*

Proof. Clearly $\mathcal{N}(T^\dagger) \subseteq \mathcal{R}(T)^\perp$ and $\mathcal{N}(T) \subseteq {}^\perp\mathcal{R}(T^\dagger)$. Let $y \in \mathcal{R}(T)^\perp$ be arbitrary. Then

$$\langle x, T^\dagger y\rangle = \langle Tx, y\rangle = 0 \quad \text{for all } x \in X.$$

By nondegeneracy of the form, $T^\dagger y = 0$ so $y \in \mathcal{N}(T^\dagger)$, thus $\mathcal{R}(T)^\perp = \mathcal{N}(T^\dagger)$. By a similar argument, ${}^\perp\mathcal{R}(T^\dagger) = \mathcal{N}(T)$. From these two equalities we get ${}^\perp(\mathcal{R}(T)^\perp) = {}^\perp\mathcal{N}(T^\dagger)$ and $({}^\perp\mathcal{R}(T^\dagger))^\perp = \mathcal{N}(T)^\perp$. From Lemma 1 we obtain the last two results of the lemma. ■

We now have the following useful lemma.

Lemma 3. *The following are true for any projection $P \in \mathcal{A}$:*
(1) $\mathcal{N}(P) = {}^\perp\mathcal{R}(P^\dagger)$*;*
(2) $\mathcal{R}(P) = {}^\perp\mathcal{N}(P^\dagger)$*;*
(3) $\mathcal{R}(P^\dagger) = \mathcal{N}(P)^\perp$*; and*
(4) $\mathcal{N}(P^\dagger) = \mathcal{R}(P)^\perp$*.*
Thus $\mathcal{R}(P)$ and $\mathcal{N}(P)$ are both $\mathcal{Y}$-closed and $\mathcal{R}(P^\dagger)$ and $\mathcal{N}(P^\dagger)$ are both $\mathcal{X}$-closed.

Proof. To prove the first two notice that both P and $I-P$ are in $\mathcal{A}$. From Lemma 1, both $\mathcal{N}(P)$ and $\mathcal{N}(I-P) = \mathcal{R}(P)$ are $\mathcal{Y}$-closed and thus Lemma 2 applies. The last two equalities use the same argument on $P^\dagger$ and $I - P^\dagger$. ■

We immediately have the following theorem.

Theorem 2.2. *Let P be a projection in $B(X)$. Then $Y = \mathcal{R}(P)^\perp \oplus \mathcal{N}(P)^\perp$ if and only if $P \in \mathcal{A}$.*

Proof. First assume that $Y = \mathcal{R}(P)^\perp \oplus \mathcal{N}(P)^\perp$. Then for any $x \in X$ and $y \in Y$ we have unique representations

$$\begin{aligned} x &= x_1 + x_2, \qquad x_1 \in \mathcal{R}(P),\ x_2 \in \mathcal{N}(P) \quad \text{and} \\ y &= y_1 + y_2, \qquad y_1 \in \mathcal{R}(P)^\perp,\ y_2 \in \mathcal{N}(P)^\perp. \end{aligned}$$

Note that $\langle x_1, y_1\rangle = 0$ and $\langle x_2, y_2\rangle = 0$. Since $\mathcal{N}(P)^\perp$ and $\mathcal{R}(P)^\perp$ are both norm-closed subspaces, we can define $Q \in B(Y)$ to be the continuous projection onto $\mathcal{N}(P)^\perp$ with nullspace $\mathcal{R}(P)^\perp$ [8, theorem IV.12.2]. Then for any $x \in X$ and $y \in Y$ and the above representations,

$$\begin{aligned} \langle x, Qy\rangle &= \langle x, y_2\rangle \\ &= \langle x_1, y_2\rangle + \langle x_2, y_2\rangle \end{aligned}$$

$$= \langle x_1, y_2 \rangle$$
$$= \langle x_1, y_1 \rangle + \langle x_1, y_2 \rangle$$
$$= \langle Px, y \rangle.$$

So $\langle Px, y\rangle = \langle x, Qy\rangle$ for all $x \in X$, $y \in Y$. Thus $P \in \mathcal{A}$ with $P^\dagger = Q$.

Now assume $P \in \mathcal{A}$. Clearly $P^\dagger \in B(Y)$ is a projection so $Y = \mathcal{N}(P^\dagger) \oplus \mathcal{R}(P^\dagger)$. By the above lemma, $\mathcal{N}(P^\dagger) = \mathcal{R}(P)^\perp$. Also, if we let $Q = I - P$, $Q \in \mathcal{A}$ so $\mathcal{R}(P^\dagger) = \mathcal{N}(Q^\dagger) = \mathcal{R}(Q)^\perp = \mathcal{N}(P)^\perp$. Thus $Y = \mathcal{R}(P)^\perp \oplus \mathcal{N}(P)^\perp$. ■

If an operator $T \in \mathcal{A}$ is invertible in $\mathcal{A}$, it is clear that $(T^{-1})^\dagger$ must equal $(T^\dagger)^{-1}$. Thus $T^\dagger$ must also be invertible in $B(Y)$. To consider when an arbitrary operator in $\mathcal{A}$ has a generalised inverse in $\mathcal{A}$ we must consider the different topologies on X and Y and how the nullspaces and ranges of T and $T^\dagger$ are related. The following result characterises the existence of generalised inverses in the Jörgens algebra and is the main result of this paper.

Theorem 2.3. *Let $T \in \mathcal{A}$. T has a generalised inverse $S \in \mathcal{A}$ if and only if*

(1) *There exist projections P and Q in $\mathcal{A}$ such that*

$$\mathcal{R}(P) = \mathcal{N}(T), \quad \mathcal{R}(Q) = \mathcal{R}(T); \quad \textit{and}$$

(2) $\mathcal{R}(T^\dagger) = \mathcal{N}(T)^\perp$.

PROOF. First assume that there exists a generalised inverse S of T such that $S \in \mathcal{A}$. Then $S^\dagger$ is a generalised inverse of $T^\dagger$. By [8, theorem IV.12.9], there exist continuous projections $P = I - ST$ and $Q = TS$ such that $\mathcal{R}(P) = \mathcal{N}(T)$ and $\mathcal{R}(Q) = \mathcal{R}(T)$. Clearly, by construction, P and Q are in $\mathcal{A}$ with $P^\dagger = I - T^\dagger S^\dagger$ and $Q^\dagger = S^\dagger T^\dagger$. By that same theorem, $T^\dagger S^\dagger$ is a projection onto $\mathcal{R}(T^\dagger)$; thus $\mathcal{N}(P^\dagger) = \mathcal{R}(T^\dagger)$. However, by Lemma 3, $\mathcal{N}(P^\dagger) = \mathcal{R}(P)^\perp = \mathcal{N}(T)^\perp$. Therefore, $\mathcal{R}(T^\dagger) = \mathcal{N}(T)^\perp$.

Conversely, suppose $\mathcal{R}(T^\dagger) = \mathcal{N}(T)^\perp$ and let P and Q be the projections as in the hypothesis (1). Let $L = \mathcal{N}(P)$ and $M = \mathcal{N}(Q)$. Then

$$X = \mathcal{N}(T) \oplus L = \mathcal{R}(T) \oplus M$$

and from Theorem 2.2,

$$Y = \mathcal{N}(T)^\perp \oplus L^\perp = \mathcal{R}(T)^\perp \oplus M^\perp.$$

By Lemma 3,

$$\mathcal{N}(P^\dagger) = \mathcal{R}(P)^\perp = \mathcal{N}(T)^\perp = \mathcal{R}(T^\dagger), \mathcal{R}(P^\dagger) = L^\perp,$$
$$\mathcal{N}(Q^\dagger) = \mathcal{R}(Q)^\perp = \mathcal{R}(T)^\perp = \mathcal{N}(T^\dagger), \text{ and } \mathcal{R}(Q^\dagger) = M^\perp.$$

Define the map $T_1 : L \to \mathcal{R}(T)$ by $T_1 x = Tx$ for all $x \in L$. Clearly T_1 is a linear, bounded, one-to-one operator onto $\mathcal{R}(T)$. Since $\mathcal{R}(T)$ is norm-closed, it is a Banach space. Thus by the Open Mapping theorem, $T_1^{-1} : \mathcal{R}(T) \to L$ exists as a bounded linear operator. Define $S : X \to X$ to be $T_1^{-1}Q$. Clearly $S \in B(X)$ and $STx = x$ for all $x \in L$. Let $x \in X$ be arbitrary. Since x can be expressed uniquely as $x = x_1 + x_2$ with $x_1 \in \mathcal{N}(T)$ and $x_2 \in L$,

$$\begin{aligned} TSTx &= TST(x_1 + x_2) \\ &= TSTx_2 \\ &= Tx_2 \\ &= T(x_1 + x_2) \\ &= Tx. \end{aligned}$$

Thus S is a generalised inverse for T.

Define $T_2 : M^{\perp} \to \mathcal{R}(T^{\dagger})$ by $T_2 y = T^{\dagger} y$. Clearly T_2 is a bounded linear operator. It is one-to-one and onto $\mathcal{R}(T^{\dagger})$ since $\mathcal{R}(T)^{\perp} = \mathcal{N}(T^{\dagger})$. Since $\mathcal{R}(T^{\dagger})$ is $\mathcal{X}$-closed, it is norm-closed so $T_2^{-1} : \mathcal{R}(T^{\dagger}) \to M^{\perp}$ exists as a bounded linear operator by the Open Mapping theorem. Define $S_2 \in B(Y)$ by $S_2 = T_2^{-1}(I - P^{\dagger})$. For any $y \in Y$ we have the unique representation $y = y_1 + y_2$ with $y_1 \in \mathcal{N}(T^{\dagger}) = \mathcal{R}(T)^{\perp}$ and $y_2 \in M^{\perp}$. Note that

$$T^{\dagger} y_2 = T_2 y_2 \quad \text{and} \quad S_2 T^{\dagger} y_2 = T_2^{-1} T^{\dagger} y_2 = T_2^{-1} T_2 y_2 = y_2.$$

Then we have for any $y \in Y$ with $y = y_1 + y_2$ as above,

$$\begin{aligned} T^{\dagger} S_2 T^{\dagger} y &= T^{\dagger} S_2 T^{\dagger} (y_1 + y_2) \\ &= T^{\dagger} S_2 T^{\dagger} y_2 \\ &= T^{\dagger} y_2 \\ &= T^{\dagger} (y_1 + y_2) \\ &= T^{\dagger} y. \end{aligned}$$

Thus S_2 is a generalised inverse of $T^{\dagger}$. For any $x \in X$ and $y \in Y$ we have

$$\begin{aligned} &x = Tx_1 + x_2, x_1 \in L, \quad x_2 \in M \quad \text{and} \\ &y = T^{\dagger} y_1 + y_2, y_1 \in M^{\perp}, \quad y_2 \in L^{\perp}. \end{aligned}$$

By Lemma 3, $\mathcal{N}(I - P^{\dagger}) = \mathcal{R}(P^{\dagger}) = L^{\perp}$ so $S_2 y_2 = T_2^{-1}(I - P^{\dagger}) y_2 = 0$. Thus

$$S_2 y = S_2 T^{\dagger} y_1 + S_2 y_2 = S_2 T^{\dagger} y_1 = y_1, \text{ since } y_1 \in M^{\perp}$$

and

$$\begin{aligned} \langle Sx, y \rangle &= \langle STx_1, y \rangle + \langle Sx_2, y \rangle \\ &= \langle x_1, T^{\dagger} y_1 \rangle \qquad (Sx_2 = 0 \text{ since } S = T_1^{-1} Q = 0 \text{ on } \mathcal{N}(Q) = M) \\ &= \langle Tx_1, y_1 \rangle \\ &= \langle Tx_1 + x_2, y_1 \rangle \qquad (\text{since } x_2 \in M, y_1 \in M^{\perp}) \\ &= \langle x, S_2 y \rangle. \end{aligned}$$

Therefore S is a generalised inverse of T in $\mathcal{A}$ with $S^\dagger = S_2$. ■

Remark 1. *With the above construction of $S = T_1^{-1}Q$, $STS = S$. This is because $TT_1^{-1}x = x$ for all $x \in R(T)$ and $QT = T$, since $\mathcal{R}(Q) = \mathcal{R}(T)$, so we have*

$$\begin{aligned} STS &= T_1^{-1}QTT_1^{-1}Q \\ &= T_1^{-1}TT_1^{-1}Q \\ &= T_1^{-1}Q \\ &= S. \end{aligned}$$

From the theorem and Lemma 3 we obtain the following corollary.

Corollary 1. *Let $T \in \mathcal{A}$ such that T has a generalised inverse $S \in \mathcal{A}$. Then $\mathcal{R}(T) = {}^{\perp}\mathcal{N}(T^\dagger)$ and $\mathcal{R}(T^\dagger) = \mathcal{N}(T)^\perp$.*

3. Banach Spaces with Bounded Inner Product

Let X be a Banach space with a bounded inner product $(\cdot,\cdot)$. For $T \in B(X)$, define T^* to be the adjoint of T with respect to the inner product. That is,

$$(Tx, y) = (x, T^*y) \text{ for all } x, y \in X.$$

Define the algebra $\mathcal{B} = \{T \in B(X) \big| \exists T^* \in B(X)\}$. This is equivalent to the algebra of all bounded linear operators on X that have bounded extensions to the Hilbert space completion of X [7]. Define a norm on the elements of $\mathcal{B}$ similar to the Jörgens algebra; that is, for $T \in \mathcal{B}$,

$$\| T \| = \max\{\| T \|_{op}, \| T^* \|_{op}\}.$$

This makes $\mathcal{B}$ a Banach *-algebra and so Moore-Penrose inverses can be discussed. If $\mathcal{B}$ is a *-algebra, $b \in \mathcal{B}$ is a *Moore-Penrose inverse* of $a \in \mathcal{B}$ if

$$aba = a, \quad bab = b, \quad (ba)^* = ba \quad \text{and} \quad (ab)^* = ab.$$

Throughout the rest of this section, $\mathcal{B}$ will denote the *-algebra above with the inner product space X and T^* will denote the adjoint of T in this algebra. As in the Jörgens algebra case we can define the space $M^\perp \subseteq X$ for a subspace M of X. For a fixed $x_o \in X$ define $\alpha_{x_o}(x) := (x, x_o)$. This is clearly a linear functional and by continuity of the inner product, $\alpha_{x_o} \in X^*$. Thus we have a weak $\mathcal{X}$-topology on X as defined in [3] and the Jörgens algebra case. All of the results about the $M^\perp$ spaces and the $\mathcal{X}$-topology in the Jörgens algebra case apply. In particular we have the following results.

Lemma 4. *The following are true for any projection $P \in \mathcal{B}$.*
(1) $\mathcal{N}(P) = \mathcal{R}(P^*)^\perp$;
(2) $\mathcal{R}(P) = \mathcal{N}(P^*)^\perp$;
(3) $\mathcal{R}(P^*) = \mathcal{N}(P)^\perp$; *and*

(4) $\mathcal{N}(P^*) = \mathcal{R}(P)^\perp$.
Thus $\mathcal{R}(P)$, $\mathcal{N}(P)$, $\mathcal{R}(P^*)$ and $\mathcal{N}(P^*)$ are all $\mathcal{X}$-closed.

Theorem 3.1. *Let P be a projection in $B(X)$. $P \in \mathcal{B}$ if and only if $X = \mathcal{R}(P)^\perp \oplus \mathcal{N}(P)^\perp$.*

Theorem 3.2. *Let $T \in \mathcal{B}$. T has a generalised inverse in $\mathcal{B}$ if and only if*

(1) *There exist projections P and Q in $\mathcal{B}$ with*
$\mathcal{R}(P) = \mathcal{N}(T)$ and $\mathcal{R}(Q) = \mathcal{R}(T)$; and
(2) *$\mathcal{R}(T^*) = \mathcal{N}(T)^\perp$.*

The proofs of these theorems are the same as the Jörgens algebra case since the only difference is that there is a sesquilinear form rather than a bilinear form.

We immediately have the following result.

Theorem 3.3. *Let $T \in \mathcal{B}$. T has a Moore-Penrose inverse in $\mathcal{B}$ if and only if*

(1) *$X = \mathcal{N}(T) \oplus \mathcal{N}(T)^\perp = \mathcal{R}(T) \oplus \mathcal{R}(T)^\perp$; and*
(2) *$\mathcal{R}(T) = \mathcal{N}(T^*)^\perp$ and $\mathcal{R}(T^*) = \mathcal{N}(T)^\perp$.*

Proof. First assume that T has a Moore-Penrose inverse $S \in \mathcal{B}$. By definition, S is a generalised inverse of T in $\mathcal{B}$ and there exist self-adjoint projections $P = I - ST$ and $Q = TS$ in $\mathcal{B}$ such that

$$\mathcal{R}(P) = \mathcal{N}(T) \quad \text{and} \quad \mathcal{R}(Q) = \mathcal{R}(T).$$

From Lemma 4,

$$\begin{aligned} \mathcal{N}(P) &= \mathcal{N}(P^*) = \mathcal{R}(P)^\perp = \mathcal{N}(T)^\perp \quad \text{and} \\ \mathcal{N}(Q) &= \mathcal{N}(Q^*) = \mathcal{R}(Q)^\perp = \mathcal{R}(T)^\perp. \end{aligned}$$

Thus $X = \mathcal{N}(T) \oplus \mathcal{N}(T)^\perp = \mathcal{R}(T) \oplus \mathcal{R}(T)^\perp$. By Theorem 3.2, $\mathcal{R}(T^*) = \mathcal{N}(T)^\perp$. By Lemma 4, $\mathcal{R}(Q) = \mathcal{R}(T)$ is $\mathcal{X}$-closed; thus $\mathcal{R}(T) = \mathcal{N}(T^*)^\perp$.

Now assume the converse. Let P be the projection onto $\mathcal{N}(T)$ along $\mathcal{N}(T)^\perp$ and Q be the projection onto $\mathcal{R}(T)$ along $\mathcal{R}(T)^\perp$. Clearly, by Theorem 3.1 both P and Q are in $\mathcal{B}$, and from Lemma 4 we have

$$\begin{aligned} \mathcal{R}(P^*) &= \mathcal{N}(P)^\perp = \mathcal{N}(T), \mathcal{N}(P^*) = \mathcal{R}(P)^\perp = \mathcal{N}(T)^\perp, \\ \mathcal{R}(Q^*) &= \mathcal{N}(Q)^\perp = \mathcal{R}(T), \mathcal{N}(Q^*) = \mathcal{R}(Q)^\perp = \mathcal{R}(T)^\perp. \end{aligned}$$

Thus $P^* = P$ and $Q^* = Q$. By Theorem 3.2, T has a generalised inverse $S \in \mathcal{B}$ such that $P = I - ST$, $Q = TS$ and $STS = S$. But $I - (ST)^* = P^* = P = I - ST$; thus $ST = (ST)^*$. Also $(TS)^* = Q^* = Q = TS$. Thus $S \in \mathcal{B}$ is a Moore-Penrose inverse of T. ■

As in the Jörgens algebra case, an operator T is invertible in $\mathcal{B}$ if and only if T and T^* are invertible in $B(X)$ [1, theorem 2.5]. Also, we say T is Fredholm with respect to $\mathcal{B}$, or $T \in \Phi_{\mathcal{B}}$, when T is invertible modulo finite rank operators in $\mathcal{B}$; i.e. there exists an operator $S \in \mathcal{B}$ and finite rank operators $F, G \in \mathcal{B}$ such that $ST = I + F$ and $TS = I + G$. This definition was discussed in [1] and shown to be a natural definition. Also, $\mathcal{B}$ can be shown to be a saturated algebra and these ideas of Fredholm operators and finite rank operators are equivalent [5, theorem 25.1]. Let $\iota(T)$ denote the Fredholm index of T, and $\Phi^0_{\mathcal{B}}$ be the set of operators $T \in \Phi_{\mathcal{B}}$ for which $\iota(T) = 0$. Also, $T \in \Phi_{\mathcal{B}}$ if and only if T and $T^* \in \Phi(X)$ and $\iota(T) + \iota(T^*) = 0$ [1, theorem 2.5]. The analogous condition [6] offers the natural definition of $\Phi_{\mathcal{A}}$.

Elements of C*-algebras that have generalised inverses also have Moore-Penrose inverses [4, theorem 6]. The proof of this result uses the symmetric property that for any element x of a C*-algebra, $I + x^*x$ is invertible. In $\mathcal{B}$ we do not necessarily have symmetry so we first need some preliminaries.

Lemma 5. *Let P be a projection in $\mathcal{B}$. Then $\mathcal{N}(P) = \mathcal{N}(PP^*P)$ and $\mathcal{N}(P^*) = \mathcal{N}(P^*PP^*)$.*

PROOF. We only need to prove the first equality. Clearly $\mathcal{N}(P) \subseteq \mathcal{N}(PP^*P)$. To prove the reverse inclusion we use the fact that for any subspace $M \subseteq X$, $M \cap M^\perp = \{0\}$. Let $PP^*Px = 0$. By Lemma 4, $P^*Px \in \mathcal{N}(P) \cap R(P^*) = \mathcal{N}(P) \cap \mathcal{N}(P)^\perp = \{0\}$ and therefore $Px \in \mathcal{N}(P^*) \cap \mathcal{R}(P) = \mathcal{R}(P)^\perp \cap \mathcal{R}(P) = \{0\}$. Thus $x \in \mathcal{N}(P)$ and we have equality. ■

Note that the above lemma is true for any $T \in \mathcal{B}$ such that $\mathcal{R}(T^*) = \mathcal{N}(T)^\perp$ and $\mathcal{R}(T) = \mathcal{N}(T^*)^\perp$ (the other two equalities from the analogue of Lemma 2 are true for any $T \in \mathcal{B}$).

Lemma 6. *Let P be a projection in $\mathcal{B}$ and $U = I + (P - P^*)^*(P - P^*) = I - (P - P^*)^2$. Then U is injective.*

PROOF. Suppose $Ux = 0$ for some $x \in X$. By definition of U,

$$x = Px + P^*x - PP^*x - P^*Px.$$

By multiplying the equation by P and P^* separately, we get both $PP^*Px = 0$ and $P^*PP^*x = 0$. By Lemma 5 $x \in \mathcal{N}(PP^*P) \cap \mathcal{N}(P^*PP^*) = N(P) \cap \mathcal{N}(P^*)$. Consequently $\mathcal{N}(U) \subseteq \mathcal{N}(P) \cap \mathcal{N}(P^*)$.

Clearly $U \in \mathcal{B}$ and $U^* = U$. So we have $\mathcal{R}(U) = \mathcal{R}(U^*) \subseteq \mathcal{N}(U)^\perp$ and $\mathcal{N}(U) = \mathcal{R}(U)^\perp$. Let $y \in \mathcal{N}(U)$. Then for all $x \in X$,

$$\begin{aligned} 0 &= (x, Uy) \\ &= (Ux, y) \\ &= (x - Px - P^*x + PP^*x + P^*Px, y) \\ &= (x, y) - (Px, y) - (P^*x, y) + (PP^*x, y) + (P^*Px, y) \end{aligned}$$

$$= (x, y) - (x, P^*y) - (x, Py) + (P^*x, P^*y) + (Px, Py)$$
$$= (x, y)$$

since $y \in \mathcal{N}(P) \cap \mathcal{N}(P^*)$. So for any $y \in \mathcal{N}(U)$, $(x, y) = (Ux, y) = 0$ for all $x \in X$. By nondegeneracy of the inner product, $y = 0$ and so U is injective. ■

Theorem 3.4. *Let $T \in \mathcal{B}$ such that T has a generalised inverse $S \in \mathcal{B}$. Let $P = ST$ and $Q = TS$. If $U = I - (P - P^*)^2$ and $V = I - (Q - Q^*)^2$ are both surjective then T has a Moore-Penrose inverse $\widehat{S}$ in $\mathcal{B}$ defined by $\widehat{S} = P^*PU^{-1}SQQ^*V^{-1}$.*

Proof. Let $S \in \mathcal{B}$ be the generalised inverse of $T \in \mathcal{B}$ and let $P = ST$ and $Q = TS$. Let $U = I + (P - P^*)^*(P - P^*) = I - (P - P^*)^2$ and $V = I + (Q - Q^*)^*(Q - Q^*) = I - (Q - Q^*)^2$. Clearly U and V are in $\mathcal{B}$ and both self adjoint. By the above lemma both U and V are injective so U and V are both invertible in $B(X)$. However, $U = U^*$ and $V = V^*$, thus U and V are both invertible in $\mathcal{B}$.

Now we apply the proof of [4, theorem 6]. The theorem states that if an element in a C*-algebra has a generalised inverse then it has a Moore-Penrose inverse in the algebra. In fact, the proof works in any *-algebra where U and V as defined above are invertible and the proof then follows. It should be noted that the Moore-Penrose inverse $\widehat{S}$ of T is constructed in the theorem as follows:

$$\widehat{S} = P^*PU^{-1}SQQ^*V^{-1}$$

where S is the generalised inverse of T in $\mathcal{B}$, $P = ST$, $Q = TS$, $U = I + (P - P^*)^*(P - P^*)$ and $V = I + (Q - Q^*)^*(Q - Q^*)$. ■

Corollary 2. *Let $T \in \mathcal{B}$ such that $T \in \Phi_{\mathcal{B}}$. Then T has a Moore-Penrose inverse in $\mathcal{B}$.*

Proof. Recall that since $T \in \Phi_{\mathcal{B}}$, $T \in \Phi(X)$, $T^* \in \Phi(X)$, and $\iota(T) + \iota(T^*) = 0$ [1, theorem 2.5]. Thus T has a generalised inverse $S \in \Phi_{\mathcal{B}}$ [6, theorem 5.16]. Clearly T^* and S^* are both in $\Phi_{\mathcal{B}}$. The projections $P = ST$, $Q = TS$, P^* and Q^* are all in $\Phi^0_{\mathcal{B}}$ [8, theorem IV.13.1] with $\mathcal{N}(P) = \mathcal{N}(T)$ and $\mathcal{R}(Q) = \mathcal{R}(T)$.

Let $U = I + (P - P^*)^*(P - P^*) = I - (P - P^*)^2$ and $V = I + (Q - Q^*)^*(Q - Q^*) = I - (Q - Q^*)^2$. As above, U and V are in $\mathcal{B}$ and both self adjoint. Clearly $PP^* \in \Phi^0_{\mathcal{B}}$. Note that $I - P$ is of finite rank since $\mathcal{R}(I - P) = \mathcal{N}(P) = \mathcal{N}(T)$. The operator $P^*P - (I - P)P^* \in \Phi^0_{\mathcal{B}}$ since $(I - P)P^*$ is of finite rank [8, theorem IV.13.4]. Using the same theorem shows that $U = I - P + (P^*P - (I - P)P^*) \in \Phi^0_{\mathcal{B}}$. A similar argument on Q shows that $V \in \Phi^0_{\mathcal{B}}$.

By Lemma 6 both U and V are injective and since both are of index zero the operators are also surjective. Thus, we apply the previous theorem to get the Moore-Penrose inverse $\widehat{S}$ of T defined by

$$\widehat{S} = P^*PU^{-1}SQQ^*V^{-1} \in \mathcal{B}.$$

■

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