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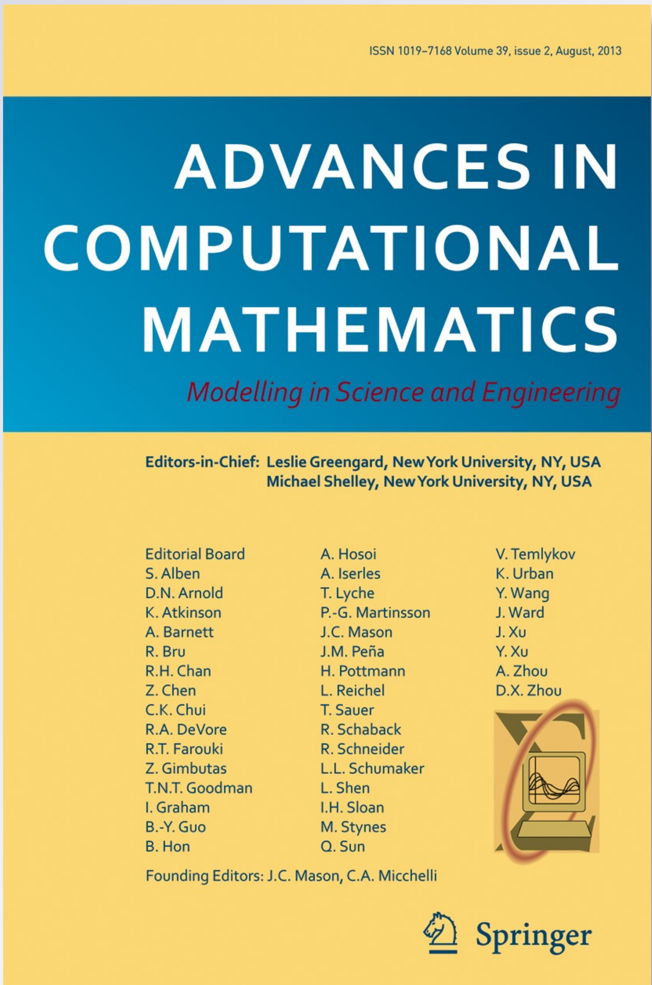

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Analytic regularity for a singularly perturbed system of reaction-diffusion equations with multiple scales

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Abstract We consider a coupled system of two singularly perturbed reaction-diffusion equations, with two small parameters $0 < \varepsilon \leq \mu \leq 1$, each multiplying the highest derivative in the equations. The presence of these parameters causes the solution(s) to have *boundary layers* which overlap and interact, based on the relative size of ε and μ. We show how one can construct full asymptotic expansions together with error bounds that cover the complete range $0 < \varepsilon \leq \mu \leq 1$. For the present case of analytic input data, we present derivative growth estimates for the terms of the asymptotic expansion that are explicit in the perturbation parameters and the expansion order.

Keywords Singular perturbation · Multiple scales · Asymptotic expansion

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1 Introduction

Singularly perturbed (SP) boundary value problems (BVPs), and their numerical approximation, have received a lot of attention in the last few decades (see, e.g., the classical texts [12, 17] on asymptotic analysis and the books [10, 11, 13], whose focus is more numerical methods for this problem class). One common feature that these problems share is the presence of *boundary layers* in the solution. In order for a numerical method, designed for the approximation of the solution to SP BVPs, to be considered *robust* it must be able to perform well, independently of the singular perturbation parameter(s). To achieve this, information about the regularity of the exact solution is utilized, and in particular, bounds on the derivatives. Such information is available in the literature for scalar SP BPVs of reaction- and convection-diffusion type in one- and two-dimensions (see, e.g., [5, 7] for scalar versions of the problem studied in the present article). For *systems* of SP BVPs, the bibliography is scarce, even in one-dimension; we mention here the pioneering paper [16] as well as [3, 4] and [1]—see also the relatively recent review article [2] and the references therein, for such results available to date. In all references quoted, however, only the first few derivatives of the various solution components are controlled. While this is sufficient for the analysis of methods of fixed order, it is insufficient for proving exponential convergence of a numerical method. Such a proof requires control of arbitrary order derivatives of the various solution components. It is the purpose of this article to begin filling this void; in particular, we provide the regularity theory for a system of two coupled SP linear reaction-diffusion equations, with two singular perturbation parameters. Our analysis is complete for the problem under consideration in that we derive full asymptotic expansions for all relevant cases of singular perturbation parameters and give explicit control of all derivatives of all terms appearing in the expansions. Even though this is a linear, one-dimensional problem, the methodology presented here can be the starting point for treating more difficult problems.

The regularity results obtained here are used in the companion communication [9] to prove, for the first time, exponential convergence of the *hp*-FEM for problems with multiple singular perturbation parameters. This exponential convergence result for the *hp*-FEM relies on mesh design principles firmly established for problems with a single singular perturbation parameter as discussed in [5–7, 14, 15]; the mathematical analysis of [9] shows that these mesh design principles extend to problems with multiple singular perturbation parameters and confirms the numerical results of [18] for the problem class under consideration here. The present paper covers all regimes of scale separation (see the cases (I)–(IV) below). While a side-by-side comparison of the asymptotic expansions for all cases is of interest in its own right, we point out that for a rigorous convergence analysis of numerical methods for the practically relevant case of small, but *fixed* singular perturbation parameters it is indeed necessary to have a regularity theory for all four cases in hand since the error term in each case is, while small, fixed for fixed singular perturbation parameters (cf. Theorem 2.2).

The rest of the paper is organized as follows: In Section 2 we present the model problem. In Sections 3–6 we develop the asymptotic expansions and present the regularity of the terms of the expansions for the different cases of relations between the singular perturbation parameters. The proofs of most of the results presented in these sections are quite tedious and therefore relegated to the preprint [8]. Finally, in Section 7 we present the results of a numerical experiment that shows how the *hp*-FEM can indeed lead to exponential convergence provided the mesh is chosen appropriately.

In what follows, the space of square integrable functions on an interval $I \subset \mathbb{R}$ will be denoted by $L^2(I)$, with associated inner product

$$\langle u, v\rangle_I := \int_I u(x)v(x)dx.$$

We will also utilize the usual Sobolev space notation $H^k(I)$ to denote the space of functions on I with $0, 1, 2, ..., k$ generalized derivatives in $L^2(I)$, equipped with norm and seminorm $\|\cdot\|_{k,I}$ and $|\cdot|_{k,I}$, respectively. For vector functions $\mathbf{U} := (u_1(x), u_2(x))^T$, we will write

$$\|\mathbf{U}\|_{k,I}^2 = \|u_1\|_{k,I}^2 + \|u_2\|_{k,I}^2.$$

We will also use the space

$$H_0^1(I) = \left\{u \in H^1(I) : u|_{\partial I} = 0\right\},$$

where ∂I denotes the boundary of I. Finally, the letters C and γ will be used to denote a generic positive constant, independent of any singular perturbation parameters and possibly having different values in each occurrence.

2 The model problem and main results

We consider the following model problem: Find a pair of functions $(u, v)^T$ such that

$$\begin{cases} -\varepsilon^2 u''(x) + a_{11}(x)u(x) + a_{12}(x)v(x) = f(x) \text{ in } I = (0,1), \\ -\mu^2 v''(x) + a_{21}(x)u(x) + a_{22}(x)v(x) = g(x) \text{ in } I = (0,1), \end{cases} \tag{2.1a}$$

along with the boundary conditions

$$u(0) = u(1) = 0\,,\;\; v(0) = v(1) = 0. \tag{2.1b}$$

With the abbreviations

$$\mathbf{U} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \mathbf{E}^{\varepsilon,\mu} := \begin{pmatrix} \varepsilon^2 & 0 \\ 0 & \mu^2 \end{pmatrix}, \quad \mathbf{A}(x) := \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} f \\ g \end{pmatrix},$$

(2.1a)–(2.1b) may also be written in the following, more compact form:

$$L_{\varepsilon,\mu}\mathbf{U} := -\mathbf{E}^{\varepsilon,\mu}\mathbf{U}''(x) + \mathbf{A}(x)\mathbf{U} = \mathbf{F}, \qquad \mathbf{U}(0) = \mathbf{U}(1) = 0. \tag{2.2}$$

The parameters $0 < \varepsilon \le \mu \le 1$ are given, as are the functions f, g, and a_{ij}, $i, j \in \{1, 2\}$, which are assumed to be analytic on $\overline{I} = [0, 1]$. Moreover we assume that there exist constants $C_f, \gamma_f, C_g, \gamma_g, C_a, \gamma_a > 0$ such that

$$\begin{cases} \left\| f^{(n)} \right\|_{L^\infty(I)} \le C_f \gamma_f^n n! \quad \forall\, n \in \mathbb{N}_0, \\ \left\| g^{(n)} \right\|_{L^\infty(I)} \le C_g \gamma_g^n n! \quad \forall\, n \in \mathbb{N}_0, \\ \left\| a_{ij}^{(n)} \right\|_{L^\infty(I)} \le C_a \gamma_a^n n! \quad \forall\, n \in \mathbb{N}_0, i, j \in \{1, 2\} \end{cases} . \tag{2.3}$$

The variational formulation of (2.1a)–(2.1b) reads: Find $\mathbf{U} := (u, v)^T \in \left[H_0^1(I)\right]^2$ such that

$$B(U, V) = F(V) \quad \forall\, \mathbf{V} := (\overline{u}, \overline{v}) \in \left[H_0^1(I)\right]^2, \tag{2.4}$$

where, with $\langle \cdot, \cdot \rangle_I$ the usual $L^2(I)$ inner product,

$$B(\mathbf{U}, \mathbf{V}) = \varepsilon^2 \left\langle u', \overline{u}' \right\rangle_I + \mu^2 \left\langle v', \overline{v}' \right\rangle_I + \langle a_{11} u + a_{12} v, \overline{u} \rangle_I + \langle a_{21} u + a_{22} v, \overline{v} \rangle_I, \tag{2.5}$$

$$F(\mathbf{V}) = \langle f, \overline{u} \rangle_I + \langle g, \overline{v} \rangle_I. \tag{2.6}$$

The matrix-valued function $\mathbf{A}$ is assumed to be pointwise positive definite, i.e., for some fixed $\alpha > 0$

$$\vec{\xi}^{\,T} \mathbf{A} \vec{\xi} \ge \alpha^2 \vec{\xi}^{\,T} \vec{\xi} \quad \forall\, \vec{\xi} \in \mathbb{R}^2 \qquad \forall x \in \overline{I}. \tag{2.7}$$

It follows that the bilinear form $B(\cdot, \cdot)$ given by (2.5) is coercive with respect to the *energy norm*

$$\|\mathbf{U}\|_{E,I}^2 \equiv \|(u, v)\|_{E,I}^2 := \varepsilon^2 |u|_{1,I}^2 + \mu^2 |v|_{1,I}^2 + \alpha^2 \left(\|u\|_{0,I}^2 + \|v\|_{0,I}^2 \right), \tag{2.8}$$

i.e.,

$$B(\mathbf{U}, \mathbf{U}) \ge \|\mathbf{U}\|_{E,I}^2 \quad \forall\, \mathbf{U} \in \left[H_0^1(I)\right]^2.$$

This, along with the continuity of $B(\cdot, \cdot)$ and $F(\cdot)$, imply the unique solvability of (2.4). We also have by the Lax-Milgram lemma, the following *a priori* estimate

$$\|\mathbf{U}\|_{E,I} \le \alpha^{-1} \sqrt{\|f\|_{0,I}^2 + \|g\|_{0,I}^2}. \tag{2.9}$$

For the development of certain asymptotic expansions, it will be convenient to observe that our assumption (2.7) implies

$$a_{kk}(x) \ge \alpha^2 \; \forall x \in \overline{I}, \quad k = 1, 2, \tag{2.10}$$

$$\det \mathbf{A}(x) \ge \alpha^2 \max\{a_{11}(x), a_{22}(x)\} \ge \alpha^4 \qquad \forall x \in \overline{I}. \tag{2.11}$$

We note that the special two-scale case when the parameters are equal, i.e. $\varepsilon = \mu$, was analyzed in [19]. In the general case considered here, there are three scales ($1 \geq \mu \geq \varepsilon$) and the regularity depends on how the scales are separated. Correspondingly, there are four cases:

(I) The "no scale separation case" which occurs when *neither* $\mu/1$ *nor* ε/μ is small.
(II) The "3-scale case" in which all scales are separated and occurs when $\mu/1$ is small *and* ε/μ is small.
(III) The first "2-scale case" which occurs when $\mu/1$ is not small *and* ε/μ is small.
(IV) The second "2-scale case" which occurs when $\mu/1$ is small *and* ε/μ is *not* small.

The concept of "small" (or "not small") mentioned above, is tied in two ways to our performing regularity theory in terms of asymptotic expansions. First, on the level of constructing asymptotic expansions, the decision which parameters are deemed small determines the ansatz to be made and thus the form of the expansion. Second, on the level of applying asymptotic expansions, the decision which parameters are deemed small depends on whether the remainder $\mathbf{R}$ resulting from the asymptotic expansion may be regarded as small.

We need to introduce some notation:

Definition 2.1

1. We say that a function w is analytic with length scale $\nu > 0$ (and analyticity parameters C_w, γ_w), abbreviated $w \in \mathcal{A}(\nu, C_w, \gamma_w)$, if

$$\|w^{(n)}\|_{L^\infty(I)} \leq C_w \gamma_w^n \max\{n, \nu^{-1}\}^n \qquad \forall n \in \mathbb{N}_0.$$

2. We say that that an entire function w is of L^∞-boundary layer type with length scale $\nu > 0$ (and analyticity parameters C_w, γ_w), abbreviated $w \in \mathcal{BL}^\infty(\nu, C_w, \gamma_w)$, if for all $x \in I$

$$|w^{(n)}(x)| \leq C_w \gamma_w^n \nu^{-n} e^{-\operatorname{dist}(x,\partial I)/\nu} \qquad \forall n \in \mathbb{N}_0.$$

3. We say that that an entire function w is of L^2-boundary layer type with length scale $\nu > 0$ (and analyticity parameters C_w, γ_w), abbreviated $w \in \mathcal{BL}^2(\nu, C_w, \gamma_w)$, if

$$\|e^{\operatorname{dist}(x,\partial I)/\nu} w^{(n)}\|_{L^2(I)} \leq C_w \nu^{1/2} \gamma_w^n \nu^{-n} \qquad \forall n \in \mathbb{N}_0.$$

All three definitions extend naturally to vector-valued functions by requiring the above bounds componentwise.

We close this section by stating the main result; the four scale separation cases (I)–(IV) listed above, correspond to the four cases listed in the following Theorem 2.2. While is the advisable to think of the decompositions given in Theorem 2.2 as being tied to certain asymptotic regimes (since only then do they provide useful information), the decompositions given in the four cases of

Theorem 2.2 are not mutually exclusive—in fact, all decompositions are valid regardless of the relation between ε and μ.

Theorem 2.2 *Let f, g, $\mathbf{A}$ satisfy (2.3), (2.7), and let $\mathbf{U}$ be the solution of (2.1). There exist constants C, b, δ, q, $\gamma > 0$ independent of $0 < \varepsilon \le \mu \le 1$ such that the following four assertions are true for the solution $\mathbf{U}$:*

(I) $\mathbf{U} \in \mathcal{A}(\varepsilon, C\varepsilon^{-1/2}, \gamma)$.

(II) *$\mathbf{U}$ can be written as $\mathbf{U} = \mathbf{W} + \widetilde{\mathbf{U}}_{BL} + \widehat{\mathbf{U}}_{BL} + \mathbf{R}$, where $\mathbf{W} \in \mathcal{A}(1, C, \gamma)$, $\widetilde{\mathbf{U}}_{BL} \in \mathcal{BL}^\infty(\delta\mu, C, \gamma)$, $\widehat{\mathbf{U}}_{BL} \in \mathcal{BL}^\infty(\delta\varepsilon, C, \gamma)$ and $\|\mathbf{R}\|_{L^\infty(\partial I)} + \|\mathbf{R}\|_{E,I} \le C\left[e^{-b/\mu} + e^{-b\mu/\varepsilon}\right]$. Additionally, the second component $\widehat{v}$ of $\widehat{\mathbf{U}}$ satisfies the sharper estimate $\widehat{v} \in \mathcal{BL}^\infty(\delta\varepsilon, C(\varepsilon/\mu)^2, \gamma)$.*

(III) *If $\varepsilon/\mu \le q$, then $\mathbf{U}$ can be written as $\mathbf{U} = \mathbf{W} + \widehat{\mathbf{U}}_{BL} + \mathbf{R}$, where $\mathbf{W} \in \mathcal{A}(\mu, C, \gamma)$, $\widehat{\mathbf{U}}_{BL} \in \mathcal{BL}^\infty(\delta\varepsilon, C, \gamma)$ and $\|\mathbf{R}\|_{L^\infty(\partial I)} + \|\mathbf{R}\|_{E,I} \le Ce^{-b/\varepsilon}$. Additionally, the second component $\widehat{v}$ of $\widehat{\mathbf{U}}$ satisfies the sharper estimate $\widehat{v} \in \mathcal{BL}^\infty(\delta\varepsilon, C(\varepsilon/\mu)^2, \gamma)$.*

(IV) *$\mathbf{U}$ can be written as $\mathbf{U} = \mathbf{W} + \widetilde{\mathbf{U}}_{BL} + \mathbf{R}$, where $\mathbf{W} \in \mathcal{A}(1, C, \gamma)$, $\widetilde{\mathbf{U}}_{BL} \in \mathcal{BL}^\infty(\delta\mu, C\sqrt{\mu/\varepsilon}, \gamma\mu/\varepsilon)$ and $\|\mathbf{R}\|_{L^\infty(\partial I)} + \|\mathbf{R}\|_{E,I} \le C(\mu/\varepsilon)^2 e^{-b/\mu}$.*

Proof This result is obtained by combining Theorems 3.1, 4.1, 5.1, and 6.1 to be found in Sections 3–6. We emphasize that some results in these theorems are slightly sharper since they analyze all terms of the asymptotic expansions, whereas Theorem 2.2 is obtained from the asymptotic expansions by suitably selecting the expansion order.

3 The no scale separation case: Case (I)

In this case neither $\mu/1$ nor ε/μ is deemed small, which means that the boundary layer effects are not very pronounced. By the analyticity of a_{ij}, f and g, we have that u and v are analytic. Moreover, we have the following theorem.

Theorem 3.1 *Let $\mathbf{U} = (u, v)^T$ be the solution to* (2.1a) and (2.1b) *with $0 < \varepsilon \le \mu \le 1$. Then, there exist constants C and $K > 0$, independent of ε and μ, such that $\mathbf{U} \in \mathcal{A}(\varepsilon, C\varepsilon^{-1/2}, K)$. Moreover, the sharper estimate*

$$\left\|\mathbf{U}^{(n)}\right\|_{0,I} \le CK^n \max\{n, \varepsilon^{-1}\}^n \quad \forall\, n \in \mathbb{N}_0, \tag{3.1}$$

holds.

Proof The L^2-based estimate (3.1) was shown in [19] for the special case $\varepsilon = \mu$. The extension to the current situation $\varepsilon \le \mu$ is straight forward. We note that the Sobolev embedding theorem in the form $\|v\|^2_{L^\infty(I)} \le C\|v\|_{L^2(I)}\|v\|_{H^1(I)}$, allows us to infer from (3.1) the assertion $\mathbf{U} \in \mathcal{A}(\varepsilon, C\varepsilon^{-1/2}, \gamma)$, for suitable C, $\gamma > 0$ independent of ε and μ. □

4 The three scale case: Case (II)

In this case all scales are separated and it occurs when *both* $\mu/1$ and ε/μ are deemed small. This is arguably the most interesting case, since boundary layers of multiple scales appear. Additionally, this case shows most clearly the general procedure for obtaining asymptotic expansions and error bounds for problems with multiple scales. Before developing the asymptotic expansion, we formulate the main result:

Theorem 4.1 *Let f, g, $\mathbf{A}$ satisfy (2.3), (2.7). The solution $\mathbf{U}$ of (2.1) can be written as $\mathbf{U} = \mathbf{W} + \widetilde{\mathbf{U}}_{BL} + \widehat{\mathbf{U}}_{BL} + \mathbf{R}$, where $\mathbf{W} \in \mathcal{A}(1, C_W, \gamma_W)$, $\widetilde{\mathbf{U}} \in \mathcal{BL}^\infty(\delta\mu, C_{BL}, \gamma_{BL})$, $\widehat{\mathbf{U}} \in \mathcal{BL}^\infty(\delta\varepsilon, C_{BL}, \gamma_{BL})$, for suitable constants C_W, C_{BL}, γ_W, γ_{BL}, $\delta > 0$ independent of μ and ε. Furthermore, $\mathbf{R}$ satisfies*

$$\|\mathbf{R}\|_{L^\infty(\partial I)} + \|L_{\varepsilon,\mu}\mathbf{R}\|_{L^\infty(I)} \le C\left[e^{-b/\mu} + e^{-b\mu/\varepsilon}\right],$$

for some constants C, $b > 0$ independent of μ and ε. In particular, $\|\mathbf{R}\|_{E,I} \le C\left[e^{-b/\mu} + e^{-b\mu/\varepsilon}\right]$.

Additionally, the second component $\widehat{v}$ of $\widehat{\mathbf{U}}_{BL}$ satisfies the sharper regularity assertion

$$\widehat{v} \in \mathcal{BL}^\infty(\delta\varepsilon, C_{BL}(\varepsilon/\mu)^2, \gamma_{BL}).$$

Proof See Section 4.5. □

Anticipating that boundary layers of length scales $O(\mu)$ and $O(\varepsilon)$ will appear at the endpoints $x = 0$ and $x = 1$, we introduce the stretched variables $\widetilde{x} = x/\mu$, $\widehat{x} = x/\varepsilon$ for the expected layers at the left endpoint $x = 0$, and variables $\widetilde{x}^R = (1 - x)/\mu$, $\widehat{x}^R = (1 - x)/\varepsilon$ for the expected behavior at the right endpoint $x = 1$. We make the following formal ansatz for the solution $\mathbf{U}$:

$$\mathbf{U} \sim \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(\frac{\mu}{1}\right)^i\left(\frac{\varepsilon}{\mu}\right)^j\left[\mathbf{U}_{ij}(x) + \widetilde{\mathbf{U}}^L_{ij}(\widetilde{x}) + \widehat{\mathbf{U}}^L_{ij}(\widehat{x}) + \widetilde{\mathbf{U}}^R_{ij}(\widetilde{x}^R) + \widehat{\mathbf{U}}^R_{ij}(\widehat{x}^R)\right], \tag{4.1}$$

where the functions $\mathbf{U}_{ij}$, $\widetilde{\mathbf{U}}^L_{ij}$, $\widehat{\mathbf{U}}^L_{ij}$ $\widetilde{\mathbf{U}}^R_{ij}$, $\widehat{\mathbf{U}}^R_{ij}$ will be determined shortly. The decomposition of Theorem 4.1 is obtained by truncating the asymptotic expansion (4.1) after a finite number of terms:

$$\mathbf{U}^M(x) := \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = \mathbf{W}_M(x) + \widehat{\mathbf{U}}^M_{BL}(\widehat{x}) + \widehat{\mathbf{V}}^M_{BL}(\widehat{x}^R) + \widetilde{\mathbf{U}}^M_{BL}(\widetilde{x}) + \widetilde{\mathbf{V}}^M_{BL}(\widetilde{x}^R) + \mathbf{R}_M(x), \tag{4.2}$$

where

$$\mathbf{W}_M(x) = \sum_{i=0}^{M_1}\sum_{j=0}^{M_2}\mu^i\left(\frac{\varepsilon}{\mu}\right)^j\begin{pmatrix} u_{ij}(x) \\ v_{ij}(x) \end{pmatrix}, \tag{4.3}$$

denotes the outer (smooth) expansion,

$$\widehat{\mathbf{U}}^M_{BL}(\widehat{x}) = \sum_{i=0}^{M_1}\sum_{j=0}^{M_2} \mu^i \left(\frac{\varepsilon}{\mu}\right)^j \begin{pmatrix} \widehat{u}^L_{ij}(\widehat{x}) \\ \widehat{v}^L_{ij}(\widehat{x}) \end{pmatrix},$$

$$\widehat{\mathbf{V}}^M_{BL}(\widehat{x}^R) = \sum_{i=0}^{M_1}\sum_{j=0}^{M_2} \mu^i \left(\frac{\varepsilon}{\mu}\right)^j \begin{pmatrix} \widehat{u}^R_{ij}(\widehat{x}^R) \\ \widehat{v}^R_{ij}(\widehat{x}^R) \end{pmatrix}, \tag{4.4}$$

denote the left and right inner (boundary layer) expansions associated with the variables $\widehat{x}, \widehat{x}^R$, respectively,

$$\widetilde{\mathbf{U}}^M_{BL}(\widetilde{x}) = \sum_{i=0}^{M_1}\sum_{j=0}^{M_2} \mu^i \left(\frac{\varepsilon}{\mu}\right)^j \begin{pmatrix} \widetilde{u}^L_{ij}(\widetilde{x}) \\ \widetilde{v}^L_{ij}(\widetilde{x}) \end{pmatrix},$$

$$\widetilde{\mathbf{V}}^M_{BL}(\widetilde{x}) = \sum_{i=0}^{M_1}\sum_{j=0}^{M_2} \mu^i \left(\frac{\varepsilon}{\mu}\right)^j \begin{pmatrix} \widetilde{u}^R_{ij}(\widetilde{x}^R) \\ \widetilde{v}^R_{ij}(\widetilde{x}^R) \end{pmatrix}, \tag{4.5}$$

denote the left and right inner (boundary layer) expansions associated with the variables $\widetilde{x}, \widetilde{x}^R$ respectively, and

$$\mathbf{R}_M(x) := \mathbf{U}(x) - \left(\mathbf{W}_M(x) + \widehat{\mathbf{U}}^M_{BL}(\widehat{x}) + \widehat{\mathbf{V}}^M_{BL}(\widehat{x}^R) + \widetilde{\mathbf{U}}^M_{BL}(\widetilde{x}) + \widetilde{\mathbf{V}}^M_{BL}(\widetilde{x}^R)\right), \tag{4.6}$$

denotes the remainder. Theorem 4.1 will be established by selecting $M_1 = O(1/\mu)$, $M_2 = O(\mu/\varepsilon)$.

4.1 Derivation of the asymptotic expansion

In order to derive equations for the functions $\mathbf{U}_{ij}, \widetilde{\mathbf{U}}^L_{ij}, \widehat{\mathbf{U}}^L_{ij}, \widetilde{\mathbf{U}}^R_{ij}, \widehat{\mathbf{U}}^R_{ij}$, the procedure is as follows: First, the ansatz (4.1) is inserted in the differential equation (2.1a), then the scales are separated and finally recursions are obtained by equating like powers of μ and ε/μ.

In order to perform the scale separation, we need to write the differential operator $L_{\varepsilon,\mu}$ in different ways on the various scales. In particular, for the $\widetilde{x}$ and the $\widehat{x}$-scales, the coefficient $\mathbf{A}$ is written, by Taylor expansion, as

$$\mathbf{A}(x) = \sum_{k=0}^{\infty} \mu^k \mathbf{A}_k \widetilde{x}^k, \qquad \mathbf{A}_k := \mathbf{A}^{(k)}(0) = \begin{pmatrix} \frac{a^{(k)}_{11}(0)}{k!} & \frac{a^{(k)}_{12}(0)}{k!} \\ \frac{a^{(k)}_{21}(0)}{k!} & \frac{a^{(k)}_{22}(0)}{k!} \end{pmatrix}, \tag{4.7}$$

$$\mathbf{A}(x) = \sum_{k=0}^{\infty} \mu^k \left(\frac{\varepsilon}{\mu}\right)^k \mathbf{A}_k \widehat{x}^k. \tag{4.8}$$

Corresponding representations are obtained for the variables $\widetilde{x}^R$ and $\widehat{x}^R$ by expanding around the right endpoint $x = 1$. Hence, the differential operator $L_{\varepsilon,\mu}$ applied to a function depending on $\widetilde{x}$ or $\widehat{x}$, takes the following form:

$$\text{on the } \widetilde{x}\text{-scale:} \qquad -\mu^{-2}\mathbf{E}^{\varepsilon,\mu}\partial_{\widetilde{x}}^2\mathbf{U}(\widetilde{x}) + \sum_{k=0}^{\infty}\mu^k\mathbf{A}_k\widetilde{x}^k\mathbf{U}(\widetilde{x}), \tag{4.9}$$

$$\text{on the } \widehat{x}\text{-scale:} \qquad -\varepsilon^{-2}\mathbf{E}^{\varepsilon,\mu}\partial_{\widehat{x}}^2\mathbf{U}(\widehat{x}) + \sum_{k=0}^{\infty}\mu^k\left(\frac{\varepsilon}{\mu}\right)^k\mathbf{A}_k\widehat{x}^k\mathbf{U}(\widehat{x}). \tag{4.10}$$

Clearly, analogous forms exist for the operator on the $\widetilde{x}^R$ and $\widehat{x}^R$ scales. We now insert the ansatz (4.1) in the differential equation (2.1a), where the differential operator $L_{\varepsilon,\mu}$ takes the form given above on the fast scales $\widetilde{x}$, $\widehat{x}$, $\widetilde{x}^R$, $\widehat{x}^R$, and we separate the scales, i.e., we view the variables x, $\widetilde{x}$, $\widehat{x}$, $\widetilde{x}^R$, $\widehat{x}^R$ as independent variables. Then, we obtain the following formal relations:

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\mu^i\left(\frac{\varepsilon}{\mu}\right)^j\left[-\mathbf{E}^{\varepsilon,\mu}\mathbf{U}_{ij}'' + \mathbf{A}(x)\mathbf{U}_{ij}\right] = \mathbf{F}, \tag{4.11}$$

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\mu^i\left(\frac{\varepsilon}{\mu}\right)^j\left[-\mu^{-2}\mathbf{E}^{\varepsilon,\mu}(\widetilde{\mathbf{U}}_{ij}^L)'' + \sum_{k=0}^{\infty}\mu^k\mathbf{A}_k\widetilde{x}^k\widetilde{\mathbf{U}}_{ij}^L\right] = 0, \tag{4.12}$$

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\mu^i\left(\frac{\varepsilon}{\mu}\right)^j\left[-\varepsilon^{-2}\mathbf{E}^{\varepsilon,\mu}(\widehat{\mathbf{U}}_{ij}^L)'' + \sum_{k=0}^{\infty}\varepsilon^k\mathbf{A}_k\widehat{x}^k\widehat{\mathbf{U}}_{ij}^L\right] = 0, \tag{4.13}$$

and two additional equations for $\widetilde{\mathbf{U}}^R$, $\widehat{\mathbf{U}}^R$, corresponding to the scales $\widetilde{x}^R$, $\widehat{x}^R$, that are completely analogous to (4.12), (4.13). We write

$$\mathbf{U}_{ij} = \begin{pmatrix} u_{ij} \\ v_{ij} \end{pmatrix}, \qquad \widetilde{\mathbf{U}}_{ij}^L = \begin{pmatrix} \widetilde{u}_{ij}^L \\ \widetilde{v}_{ij}^L \end{pmatrix}, \qquad \widehat{\mathbf{U}}_{ij}^L = \begin{pmatrix} \widehat{u}_{ij}^L \\ \widehat{v}_{ij}^L \end{pmatrix}, \tag{4.14}$$

and equate like powers of μ and ε/μ in (4.11), (4.12), (4.13) to get the following recursions:

$$-\begin{pmatrix} u_{i-2,j-2}'' \\ v_{i-2,j}'' \end{pmatrix} + \mathbf{A}(x)\mathbf{U}_{ij} = \mathbf{F}_{ij}, \tag{4.15a}$$

$$-\begin{pmatrix} (\widetilde{u}_{i,j-2}^L)'' \\ (\widetilde{v}_{i,j}^L)'' \end{pmatrix} + \sum_{k=0}^{i}\mathbf{A}_k\widetilde{x}^k\widetilde{\mathbf{U}}_{i-k,j}^L = 0, \tag{4.15b}$$

$$-\begin{pmatrix} (\widehat{u}_{i,j}^L)'' \\ (\widehat{v}_{i,j+2}^L)'' \end{pmatrix} + \sum_{k=0}^{\min\{i,j\}}\mathbf{A}_k\widehat{x}^k\widehat{\mathbf{U}}_{i-k,j-k}^L = 0, \tag{4.15c}$$

where we adopt the convention that if a function appears with a negative subscript, then it is assumed to be zero. Furthermore, we set

$$\mathbf{F}_{00} = \begin{pmatrix} f \\ g \end{pmatrix}, \qquad \mathbf{F}_{ij} = 0 \quad \text{if } (i,j) \neq (0,0).$$

The procedure so far leads to a formal solution $\mathbf{U}$ of the differential equation (2.1a); further boundary conditions are imposed in order to conform to the boundary conditions (2.1b), namely,

$$\mathbf{U}_{ij}(0) + \widetilde{\mathbf{U}}_{ij}^{L}(0) + \widehat{\mathbf{U}}_{ij}^{L}(0) = 0, \text{ plus decay conditions for } \widetilde{\mathbf{U}}_{ij}^{L}, \widehat{\mathbf{U}}_{ij}^{L} \text{ at } +\infty, \tag{4.15d}$$

with analogous conditions at the right endpoint $x = 1$, which couple $\mathbf{U}_{ij}$, $\widetilde{\mathbf{U}}_{ij}^{R}$, and $\widehat{\mathbf{U}}_{ij}^{R}$.

4.2 Analysis of the functions $\mathbf{U}_{ij}$

Since the matrix $\mathbf{A}(x)$ is invertible $\forall x \in I$, (4.15a) may be solved for any i, j yielding

$$\begin{pmatrix} u_{0,0} \\ v_{0,0} \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} f \\ g \end{pmatrix}, \tag{4.16}$$

and for $(i, j) \neq (0, 0)$

$$\begin{pmatrix} u_{ij} \\ v_{ij} \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} u''_{i-2,j-2} \\ v''_{i-2,j} \end{pmatrix}, \tag{4.17}$$

with, as mentioned above,

$$u_{ij} = 0\ ,\ \ v_{ij} = 0 \text{ if } i < 0 \text{ or } j < 0.$$

Note that (4.17) gives all the cases $(i, 0)$ and $(0, j)$ because the right-hand side in (4.17) is known. Moreover, for each j, (4.17) allows us to compute u_{ij}, v_{ij} $\forall$ i, thus (4.16) and (4.17) uniquely determine u_{ij}, v_{ij} $\forall$ i, j.

We have the following lemma concerning the regularity of the functions $\mathbf{U}_{ij}$:

Lemma 4.2 *Let f, g, and $\mathbf{A}$ satisfy (2.3) and (2.7). Let u_{ij}, v_{ij} be the solutions of (4.16), (4.17). Then there exist positive constants C_S and K and a complex neighborhood $G \subset \mathbb{C}$ of the closed interval $\overline{I}$, independent of i and j, such that*

$$u_{ij} = v_{ij} = 0 \ \forall\ j > i, \tag{4.18}$$

$$u_{ij} = v_{ij} = 0 \qquad \text{if } i \text{ or } j \text{ is odd}, \tag{4.19}$$

$$\left|u_{ij}(z)\right| + \left|v_{ij}(z)\right| \leq C_S \delta^{-i} K^i i^i \ \forall\ z \in G_\delta := \{z \in G\colon\ \operatorname{dist}(z, \partial G) > \delta\}. \tag{4.20}$$

Proof The proof is by induction on i, where the estimate (4.20) follows by arguments of the type worked out in the proof of [5, Lemma 2]. □

4.3 Analysis of $\widetilde{\mathbf{U}}_{ij}^{L}$, $\widehat{\mathbf{U}}_{ij}^{L}$

4.3.1 Regularity of the functions $\widetilde{\mathbf{U}}_{ij}^{L}$ and $\widehat{\mathbf{U}}_{ij}^{L}$

We turn our attention to equations (4.15b) and (4.15c), which, after introducing appropriate boundary conditions, determine $\widetilde{u}_{ij}^{L}$, $\widetilde{v}_{ij}^{L}$ and $\widehat{u}_{ij}^{L}$, $\widehat{v}_{ij}^{L}$, respectively. These equations turn out to be systems of differential-algebraic equations (DAEs); however, their structure is such that the algebraic side constraint of the DAE can be eliminated explicitly and, additionally, we will be able to solve for four scalar functions sequentially instead of having to consider the coupled system. We recall that the functions $\mathbf{U}_{ij} = (u_{ij}, v_{ij})^T$ have been defined and studied in Section 4.2.

These equations may be solved by induction on j and i. For $j = 0$, we solve (4.15b) for any $(i, 0)$ by first solving for $\widetilde{u}^L_{i,0}$ and inserting it into the equation for $\widetilde{v}^L_{i,0}$. We have from (4.15b, 1st eqn)

$$\widetilde{u}^L_{i,0} = -\frac{a_{12}(0)}{a_{11}(0)}\widetilde{v}^L_{i,0} - \frac{1}{a_{11}(0)}\sum_{k=1}^{i}\frac{\widetilde{x}^k}{k!}\left[a^{(k)}_{11}(0)\widetilde{u}^L_{i-k,0} + a^{(k)}_{12}(0)\widetilde{v}^L_{i-k,0}\right], \tag{4.21}$$

which, upon inserted into (4.15b, 2nd eqn) gives

$$\begin{aligned} &-\left(\widetilde{v}^L_{i,0}\right)'' + \frac{a_{11}(0)a_{22}(0) - a_{21}(0)a_{12}(0)}{a_{11}(0)}\widetilde{v}^L_{i,0} \\ &= \sum_{k=1}^{i}\frac{\widetilde{x}^k}{k!}\left[\left(\frac{a_{21}(0)}{a_{11}(0)}a^{(k)}_{11}(0) - a^{(k)}_{21}(0)\right)\widetilde{u}^L_{i-k,0}\right. \\ &\qquad \left. + \left(\frac{a_{21}(0)}{a_{11}(0)}a^{(k)}_{12}(0) - a^{(k)}_{22}(0)\right)\widetilde{v}^L_{i-k,0}\right]. \end{aligned} \tag{4.22a}$$

The above second order differential equation is now posed as an equation in $(0, \infty)$ and supplemented with the two “boundary” conditions

$$\widetilde{v}^L_{i,0}(0) = -v_{i,0}(0), \qquad \widetilde{v}^L_{i,0}(\widetilde{x}) \to 0 \qquad \text{as } \widetilde{x} \to \infty. \tag{4.22b}$$

So, solving (4.22) gives us $\widetilde{v}^L_{i,0}$ and then from (4.21) we get $\widetilde{u}^L_{i,0}$. Inductively, we obtain $\widetilde{v}^L_{i,0}$ and $\widetilde{u}^L_{i,0}$ for all $i \geq 0$.

Next, we set

$$\widehat{v}^L_{i,0} = \widehat{v}^L_{i,1} = 0, \tag{4.23}$$

and we solve with $j = 0$ (4.15c, 1st eqn) for $\widehat{u}^L_{i,0}$ (using $\widehat{v}^L_{i,0} = 0$) with boundary conditions from $u_{i,0}$:

$$\begin{cases} -\left(\widehat{u}^L_{i,0}\right)'' + a_{11}(0)\widehat{u}^L_{i,0} = 0 \\ \widehat{u}^L_{i,0}(0) = -u_{i,0}, \qquad \widehat{u}^L_{i,0}(\widehat{x}) \to 0 \quad \text{for } \widehat{x} \to \infty. \end{cases} \tag{4.24}$$

Then, we solve (4.15c, 2nd eqn) for $\widehat{v}^L_{i,2}$:

$$\widehat{v}^L_{i,2}(z) = \int_z^\infty \int_t^\infty a_{21}(0)\widehat{u}^L_{i,0}(\tau)\,d\tau\,dt. \tag{4.25}$$

In general, assume we have performed the previous steps and we have determined $\widetilde{u}^L_{i,j}$, $\widetilde{v}^L_{i,j}$, $\widehat{u}^L_{i,j}$, $\widehat{v}^L_{i,j+2}$ for all $i \geq 0$, and second index up to j. To obtain the corresponding functions (with j replaced by $j+1$) we proceed analogously. We first solve (4.15b, 1st eqn) for $\widetilde{u}^L_{i,j+1}$,

$$\widetilde{u}^L_{i,j+1} = -\frac{a_{12}(0)}{a_{11}(0)}\widetilde{v}^L_{i,j+1} + \frac{\left(\widetilde{u}^L_{i,j-1}\right)''}{a_{11}(0)} - \frac{1}{a_{11}(0)}\sum_{k=1}^{i}\frac{\widetilde{x}^k}{k!}\left[a^{(k)}_{11}(0)\widetilde{u}^L_{i-k,j+1} + a^{(k)}_{12}(0)\widetilde{v}^L_{i-k,j+1}\right], \tag{4.26}$$

and plug it into (4.15b, 2nd eqn):

$$-\left(\widetilde{v}^L_{i,j+1}\right)'' + \frac{a_{11}(0)a_{22}(0) - a_{21}(0)a_{12}(0)}{a_{11}(0)}\widetilde{v}^L_{i,j+1} = -\frac{a_{21}(0)}{a_{11}(0)}\left(\widetilde{u}^L_{i,j-1}\right)'' + \sum_{k=1}^{i}\frac{\widetilde{x}^k}{k!}\left[\left(\frac{a_{21}(0)}{a_{11}(0)}a^{(k)}_{11}(0) - a^{(k)}_{21}(0)\right)\widetilde{u}^L_{i-k,j+1} + \left(\frac{a_{21}(0)}{a_{11}(0)}a^{(k)}_{12}(0) - a^{(k)}_{22}(0)\right)\widetilde{v}^L_{i-k,j+1}\right]. \tag{4.27a}$$

The second order ODE, (4.27a), is supplemented with the boundary conditions

$$\widetilde{v}^L_{i,j+1}(0) = -\left(v_{i,j+1}(0) + \widehat{v}^L_{i,j+1}(0)\right), \qquad \widetilde{v}^L_{i,j+1}(\widetilde{x}) \to 0 \quad \text{for } \widetilde{x} \to \infty. \tag{4.27b}$$

This gives us $\widetilde{v}^L_{i,j+1}$ and in turn $\widetilde{u}^L_{i,j+1}$ from (4.26).

Next, we solve (4.15c, 1st eqn) for $\widehat{u}^L_{i,j+1}$ with boundary conditions from $u_{i,j+1}$ and $\widetilde{u}^L_{i,j+1}$:

$$-\left(\widehat{u}^L_{i,j+1}\right)'' + a_{11}(0)\widehat{u}^L_{i,j+1} = a_{12}(0)\widehat{v}^L_{i,j+1} - \sum_{k=1}^{\min\{i,j+1\}}\frac{\widehat{x}^k}{k!}\left(a^{(k)}_{11}(0)\widehat{u}^L_{i-k,j+1-k} - a^{(k)}_{12}(0)\widehat{v}^L_{i-k,j+1-k}\right), \tag{4.28a}$$

$$\widehat{u}^L_{i,j+1}(0) = -\left(u_{i,j+1}(0) + \widetilde{u}^L_{i,j+1}(0)\right), \tag{4.28b}$$

$$\widehat{u}^L_{i,j+1}(\widehat{x}) \to 0 \quad \text{for } \widehat{x} \to \infty. \tag{4.28c}$$

Finally, we solve (4.15c, 2nd eqn) for $\widehat{v}^L_{i,j+3}$:

$$\widehat{v}^L_{i,j+3}(z) = \sum_{k=0}^{\min\{i,j+1\}}\frac{1}{k!}\int_z^\infty\int_t^\infty \tau^k\left\{a^{(k)}_{21}(0)\widehat{u}^L_{i-k,j+1-k}(\tau) + a^{(k)}_{22}(0)\widehat{v}^L_{i-k,j+1-k}(\tau)\right\}d\tau\, dt. \tag{4.29}$$

The following theorem establishes the regularity of the functions $\widetilde{u}^L_{i,j}$, $\widetilde{v}^L_{i,j}$, $\widehat{u}^L_{i,j}$, $\widehat{v}^L_{i,j}$.

Theorem 4.3 *Assume that f, g, and $\mathbf{A}$ satisfy (2.3) and (2.7). Let $\widetilde{u}^L_{i,j}$, $\widetilde{v}^L_{i,j}$, $\widehat{u}^L_{i,j}$, $\widehat{v}^L_{i,j}$ be defined recursively as above, i.e., they solve (4.21), (4.22), (4.24), (4.23), (4.25) for the case $j = 0$ and (4.26), (4.27), (4.28), (4.29) for $j \geq 1$. Set*

$$\overline{a} := \max\left\{a_{11}(0), \frac{a_{11}(0)a_{22}(0) - a_{21}(0)a_{12}(0)}{a_{11}(0)}\right\} > 0,$$

$$\underline{a} := \min\left\{a_{11}(0), \frac{a_{11}(0)a_{22}(0) - a_{21}(0)a_{12}(0)}{a_{11}(0)}\right\} > 0,$$

$$\operatorname{Exp}(z) := \begin{cases} e^{-\underline{a}\operatorname{Re}(z)} & \text{for } \operatorname{Re}(z) \geq 0 \\ e^{-\overline{a}\operatorname{Re}(z)} & \text{for } \operatorname{Re}(z) < 0. \end{cases}$$

Then the functions $\widetilde{u}^L_{ij}$, $\widetilde{v}^L_{ij}$, $\widehat{u}^L_{ij}$, $\widehat{v}^L_{ij}$ are entire functions, and there exist positive constants $C, C', K > 0$ independent of ε and μ such that

$$\left|\widetilde{\mathbf{U}}^L_{i,j}(z)\right| \leq CK^{i+j}\left(C'(i+j) + |z|\right)^{2(i+j)} \frac{1}{(i+j)!} \operatorname{Exp}(z), \quad (4.30)$$

$$\left|\widehat{u}^L_{i,j}(z)\right| + \left|\widehat{v}^L_{i,j+2}(z)\right| \leq CK^{i+j}\left(C'(i+j) + |z|\right)^{2(i+j)} \frac{1}{(i+j)!} \operatorname{Exp}(z), \quad (4.31)$$

while $\widehat{v}^L_{i,0} = \widehat{v}^L_{i,1} \equiv 0$.

Proof The proof is by induction on j and i and resembles structurally the procedure in [6, Section 7.3]. After establishing the claims for the base cases $(i, j) = (0, 0)$, $(i, j) \in \{0\} \times \mathbb{N}$, $(i, j) \in \mathbb{N} \times \{0\}$, one shows it by induction on j with induction arguments on i as part of the induction argument in j. The structure of the equations defining $\widetilde{u}^L_{ij}, \widetilde{v}^L_{ij}, \widehat{u}^L_{ij}, \widehat{v}^L_{ij}$, is such that one can proceed successively in the induction argument on j by providing estimates for $\widetilde{u}^L_{ij}$, $\widetilde{v}^L_{ij}$, $\widehat{u}^L_{ij}, \widehat{v}^L_{ij}$ in turn. □

We conclude this section by showing that the boundary layer functions are in fact entire:

Corollary 4.4 *The functions $\widetilde{\mathbf{U}}^L_{i,j}$, and $\widehat{\mathbf{U}}^L_{i,j}$ are entire functions, and there exist constants $C, \gamma_1, \gamma_2, \beta > 0$, independent of i, j, n, such that for all $x \geq 0$*

$$|\widetilde{u}^{(n)}_{i,j}(x)| + |\widetilde{v}^{(n)}_{i,j}(x)| + |\widehat{u}^{(n)}_{i,j}(x)| + |\widehat{v}^{(n)}_{i,j+2}(x)| \leq Ce^{-\beta x}\gamma_1^{i+j}(i+j)^{i+j}\gamma_2^n \qquad \forall n \in \mathbb{N}_0.$$

Proof The result follows from Theorem 4.3 and Cauchy's Integral Theorem for derivatives. □

Finally, by a summation argument, we can show that the functions $\widetilde{\mathbf{U}}^M_{BL}$ and $\widehat{\mathbf{U}}^M_{BL}$ defined in (4.4) and (4.5) are indeed of the boundary layer type given in Definition 2.1 if the expansion orders M_1 and M_2 are selected not larger than $O(1/\mu)$ and $O(\mu/\varepsilon)$, respectively.

Theorem 4.5 *Let f, g, $\mathbf{A}$ satisfy (2.3), (2.7). There exist constants C, δ, γ, $K > 0$ independent of ε and μ, such that under the assumptions $\mu(M_1+1)K \le 1$ and $\varepsilon/\mu(M_2+1)K \le 1$, there holds for the boundary layer functions $\widetilde{\mathbf{U}}_{BL}^M$ and $\widehat{\mathbf{U}}_{BL}^M$ of (4.4) and (4.5), viewed as functions of x via the changes of variables $\widetilde{x} = x/\mu$ and $\widehat{x} = x/\varepsilon$, that $\widetilde{\mathbf{U}}_{BL}^M \in \mathcal{BL}^\infty(\delta\mu, C, \gamma)$ and $\widehat{\mathbf{U}}_{BL}^M \in \mathcal{BL}^\infty(\delta\varepsilon, C, \gamma)$. Furthermore, the second component $\widehat{v}$ of $\widehat{\mathbf{U}}_{BL}^M$ satisfies the stronger assertion $\widehat{v} \in \mathcal{BL}^\infty(\delta\varepsilon, C(\varepsilon/\mu)^2, \gamma)$.*

Proof We do not work out the details here since structurally similar arguments can be found, for example, in the proofs of [5, Theorem 3] or [6, Theorems 7.2.2, 7.3.3]. Essentially, by inserting the bounds of Corollary 4.4 in the sums defining $\widetilde{\mathbf{U}}_{BL}^M$, $\widehat{\mathbf{U}}_{BL}^M$ and using the conditions $\mu(M_1+1)K \le 1$ and $\varepsilon/\mu(M_2+1)K \le 1$ for K sufficiently large one obtains upper estimates in the form of (convergent) geometric series. We point out that the sharper estimates for the second component $\widehat{v}$ of $\widehat{\mathbf{U}}_{BL}^M$ stems from the fact that $\widehat{v}_{i,0} = \widehat{v}_{i,1} = 0$. □

4.4 Remainder estimates

In this section, we analyze $\mathbf{R}_M$ defined by the decomposition (4.6). This is done by estimating the residual $L_{\varepsilon,\mu}\mathbf{R}_M$ and then appealing to the stability estimate (2.9). We will estimate $L_{\varepsilon,\mu}\mathbf{W}_M - \mathbf{F}$, $L_{\varepsilon,\mu}\widetilde{\mathbf{U}}_{BL}^M$, and $L_{\varepsilon,\mu}\widehat{\mathbf{U}}_{BL}^M$.

Theorem 4.6 *Let f, g, $\mathbf{A}$ satisfy (2.3) and (2.7). Let $\mathbf{U}$ be the solution to the problem (2.1). Then, there exist γ, $C > 0$ depending only on f, g, and $\mathbf{A}$, such that the following is true: If M_1, $M_2 \in \mathbb{N}$ are such that*

$$\mu(M_1+1)\gamma \le 1 \qquad \textit{and} \qquad \frac{\varepsilon}{\mu}(M_2+1)\gamma \le 1, \tag{4.32}$$

then the functions $\mathbf{W}_M$, $\widetilde{\mathbf{U}}_M$, and $\widehat{\mathbf{U}}_M$, given by (4.3), (4.4), and (4.5), satisfy the following:

$$\left\| L_{\varepsilon,\mu}\left(\mathbf{U} - \mathbf{W}_M\right)\right\|_{L^\infty(I)} \le C\mu^2 \left[\frac{1}{\mu - \varepsilon}\left(\mu M_1 \gamma\right)^{M_1} + \left(\frac{\varepsilon}{\mu}\right)^{M_2+2}\right],$$

$$\left|L_{\varepsilon,\mu}\widetilde{\mathbf{U}}_M(\widetilde{x})\right| \le Ce^{-\underline{a}\widetilde{x}/2}\begin{cases}(\mu(M_1+1)\gamma)^{M_1+1} + \left(\frac{\varepsilon}{\mu}(M_2+1)\gamma\right)^{M_2+1} & \text{if } \gamma\widetilde{x}\mu \le 1 \\ 1 & \text{if } \gamma\widetilde{x}\mu > 1\end{cases},$$

$$\left|L_{\varepsilon,\mu}\widehat{\mathbf{U}}_M(\widehat{x})\right| \le Ce^{-\underline{a}\widehat{x}/2}\begin{cases}(\mu(M_1+1)\gamma)^{M_1} + \left(\frac{\varepsilon}{\mu}(M_2+1)\gamma\right)^{M_2-1} & \text{if } \gamma\widehat{x}\varepsilon \le 1 \\ 1 & \text{if } \gamma\widehat{x}\varepsilon > 1\end{cases}.$$

Furthermore, the remainder $\mathbf{R}_M = \mathbf{U} - \left[\mathbf{W}_M + \widetilde{\mathbf{U}}_{BL}^M + \widetilde{\mathbf{V}}_{BL}^M + \widehat{\mathbf{U}}_{BL}^M + \widehat{\mathbf{V}}_{BL}^M\right]$ *satisfies at the endpoints of I*

$$\|\mathbf{R}_M\|_{L^\infty(\partial I)} \le C\left[e^{-b/\varepsilon} + e^{-b\mu/\varepsilon}\right].$$

for some $b > 0$ *independent of* ε *and* μ.

Proof By and large, these estimates follow from the construction of the asymptotic expansions. An issue worth commenting on is the dichotomy $\mu\widetilde{x}\gamma \le 1$ and $\mu\widetilde{x}\gamma > 1$ (and the corresponding one for $\varepsilon\widehat{x}\gamma$): the ultimate reason for this is that the Taylor expansion of the coefficient $\mathbf{A}$ near the endpoints $x = 0$ and $x = 1$ converges only in a sufficiently small neighborhood of these endpoints. Outside such neighborhoods, it is merely the exponential decay of the functions $\widetilde{\mathbf{U}}_{BL}^M$ and $\widehat{\mathbf{U}}_{BL}^M$ that can be exploited. □

4.5 Proof of Theorem 4.1

The key idea is to select the expansion orders M_1 and M_2 so as minimize the residual $L_{\varepsilon,\mu}\mathbf{R}_M$, which is achieved with the choices $M_1 = O(1/\mu)$ and $M_2 = O(\mu/\varepsilon)$. More precisely, from the estimates for the residual $L_{\varepsilon,\mu}\mathbf{R}_M$ of Theorems 4.6, we infer the existence of $q > 0$ such that under the assumption

$$\mu \le q \qquad \text{and} \qquad \frac{\varepsilon}{\mu} \le q,$$

the choice $M_1 \approx \lambda 1/\mu$ and $M_2 \approx \lambda\mu/\varepsilon$ for sufficiently small $\lambda > 0$ (but independent of ε and μ) yields

$$\|\mathbf{R}_M\|_{L^\infty(\partial I)} + \|L_{\varepsilon,\mu}\mathbf{R}_M\|_{L^\infty(I)} \le C\left[e^{-b/\mu} + e^{-b\mu/\varepsilon}\right],$$

where $C, b > 0$ are independent of μ and ε. Hence, by stability we get that $\|\mathbf{R}_M\|_{E,I} \le C\left[e^{-b/\mu} + e^{-b\mu/\varepsilon}\right]$. The estimates for the boundary layer contributions $\widetilde{\mathbf{U}}_{BL}^M$ and $\widehat{\mathbf{U}}_{BL}^M$, follow from Theorem 4.5.

It remains to formulate a decomposition of $\mathbf{U}$ for the case that $\mu \ge q$ or $\varepsilon/\mu \ge q$. In this case, we have that $e^{-b/\mu} + e^{-b\mu/\varepsilon}$ is $O(1)$. Given that $\|\mathbf{U}\|_{E,I} = O(1)$, the trivial decomposition $\mathbf{U} = 0 + 0 + 0 + \mathbf{R}_M$ provides the desired splitting.

5 The first two scale case: Case (III)

In this case it is assumed that $\mu/1$ is *not* deemed small *but* ε/μ is deemed small. In other words, only the first equation in our model problem (2.1) is viewed as being singularly perturbed. The main result of this section is the following regularity assertion. We stress that we track the dependence on the parameter μ, for example, in the assertion $\mathbf{W} \in \mathcal{A}(\mu, C_W, \gamma_W)$, which captures how derivatives of the "smooth" part $\mathbf{W}$ of the expansion depend on μ.

Theorem 5.1 *Let f, g, and $\mathbf{A}$ satisfy (2.3) and (2.7). There exist constants C_W, γ_W, C_{BL}, γ_{BL}, δ, b, $q > 0$, independent of ε and μ, such that for $\varepsilon/\mu \le q$ the solution $\mathbf{U}$ of (2.1) can be written as $\mathbf{U} = \mathbf{W} + \widehat{\mathbf{U}}_{BL} + \mathbf{R}$, where $\mathbf{W} \in \mathcal{A}(\mu, C_W, \gamma_W)$, $\widehat{\mathbf{U}}_{BL} \in \mathcal{BL}^\infty(\delta\varepsilon, C_{BL}, \gamma_{BL})$. Furthermore, $\mathbf{R}$ satisfies*

$$\|\mathbf{R}\|_{L^\infty(\partial I)} + \|L_{\varepsilon,\mu}\mathbf{R}\|_{L^\infty(I)} \le Ce^{-b/\varepsilon}.$$

In particular, $\|\mathbf{R}\|_{E,I} \le Ce^{-b/\varepsilon}$.

Additionally, the second component $\widehat{v}$ of $\widehat{\mathbf{U}}_{BL}$ satisfies the sharper regularity assertion

$$\widehat{v} \in \mathcal{BL}^\infty(\delta\varepsilon, C_{BL}(\varepsilon/\mu)^2, \gamma_{BL}).$$

Proof See Section 5.3. □

We employ again the notation of the outset of Section 4 concerning the stretched variables $\widehat{x} = x/\varepsilon$ and $\widehat{x}^R = (1-x)/\varepsilon$, and make the formal ansatz

$$\mathbf{U}(x) \sim \sum_{i=0}^{\infty} \left(\frac{\varepsilon}{\mu}\right)^i \left[\mathbf{U}_i(x) + \widehat{\mathbf{U}}_i^L(\widehat{x}) + \widehat{\mathbf{U}}_i^R(\widehat{x}^R)\right]. \tag{5.1}$$

We proceed as in Section 4 by inserting the ansatz (5.1) in the differential (2.1a), separating the slow (x) and the fast ($\widehat{x}$ and $\widehat{x}^L$) variables and equating like powers of ε/μ. We also recall that the differential operator $L_{\varepsilon,\mu}$ takes the form (4.10) on the $\widehat{x}$-scale. The separation of the slow and the fast variables leads to the following formal equations:

$$\sum_{i=0}^{\infty} \left(\frac{\varepsilon}{\mu}\right)^i \left(-\mathbf{E}^{\varepsilon,\mu}\mathbf{U}_i'' + \mathbf{A}(x)\mathbf{U}_i\right) = \mathbf{F} = \begin{pmatrix} f \\ g \end{pmatrix},$$

$$\sum_{i=0}^{\infty} \left(\frac{\varepsilon}{\mu}\right)^i \left(-\varepsilon^{-2}\mathbf{E}^{\varepsilon,\mu}(\widehat{\mathbf{U}}_i^L)'' + \sum_{k=0}^{\infty} \mu^k \left(\frac{\varepsilon}{\mu}\right)^k \widehat{x}^k \mathbf{A}_k \widehat{\mathbf{U}}_i^L\right) = 0, \tag{5.2a}$$

and an analogous equation for $\widehat{\mathbf{U}}_i^R$. Writing again

$$\widehat{\mathbf{U}}_i = \begin{pmatrix} u_i \\ v_i \end{pmatrix}, \qquad \widehat{\mathbf{U}}_i^L = \begin{pmatrix} \widehat{u}_i^L \\ \widehat{v}_i^L \end{pmatrix},$$

we obtain from (5.2), by equating like powers of ε/μ:

$$-\mu^2 u''_{i-2} + a_{11}u_i + a_{12}v_i = f_i, \tag{5.3a}$$

$$-\mu^2 v''_i + a_{21}u_i + a_{22}v_i = g_i, \tag{5.3b}$$

$$-(\widehat{u}_i^L)'' + \sum_{k=0}^{i} \mu^k \widehat{x}^k \left(\frac{a_{11}^{(k)}(0)}{k!} \widehat{u}_{i-k}^L + \frac{a_{12}^{(k)}(0)}{k!} \widehat{v}_{i-k}^L \right) = 0, \tag{5.3c}$$

$$-(\widehat{v}_{i+2}^L)'' + \sum_{k=0}^{i} \mu^k \widehat{x}^k \left(\frac{a_{21}^{(k)}(0)}{k!} \widehat{u}_{i-k}^L + \frac{a_{22}^{(k)}(0)}{k!} \widehat{v}_{i-k}^L \right) = 0, \tag{5.3d}$$

where we employed the definition of $\mathbf{A}_k$, the notation $f_0 = f$, $g_0 = g$ as well as $f_i = g_i = 0$ for $i > 0$ and the convention that function with negative subscripts are zero. Corresponding equations are satisfied by the functions $\widehat{u}_i^R$ and $\widehat{v}_i^R$. The boundary condition (2.1b) is accounted for by stipulating $\mathbf{U}_i(0) + \widehat{\mathbf{U}}_i^L(0) = 0$ and $\mathbf{U}_i(1) + \widehat{\mathbf{U}}_i^R(0) = 0$ for all $i \geq 0$ and suitable decay conditions for $\widehat{\mathbf{U}}_i^L$ as $\widehat{x} \to \infty$ and, correspondingly, for $\widehat{\mathbf{U}}_i^R$ as $\widehat{x}^R \to \infty$. Rearranging the above equations and incorporating these boundary conditions, we get a recursion of systems of DAEs, in which the algebraic constraints can be accounted for explicitly. We obtain for $i = 0, 1, 2, \ldots$:

$$\begin{cases} -\mu^2 v''_i + \frac{(a_{22}a_{11} - a_{12}a_{21})}{a_{11}} v_i = g_i - \frac{a_{21}}{a_{11}} \left(f_i + \mu^2 u''_{i-2} \right) \\ v_i(0) = -\widehat{v}_i^L(0)\ ,\ \ v_i(1) = -\widehat{v}_i^R(0) \end{cases}, \tag{5.4}$$

$$u_i = \frac{1}{a_{11}} \left(f_i + \mu^2 u''_{i-2} - a_{12}v_i \right), \tag{5.5}$$

$$\begin{cases} -\left(\widehat{u}_i^L\right)'' + a_{11}(0)\widehat{u}_i^L = -\sum_{k=1}^{i} \left(\frac{a_{11}^{(k)}(0)}{k!} \mu^k \widehat{x}^k \widehat{u}_{i-k}^L + \frac{a_{12}^{(k)}(0)}{k!} \widehat{x}^k \mu^k \widehat{v}_{i-k}^L \right) - a_{12}(0)\widehat{v}_i^L \\ \widehat{u}_i^L(0) = -u_i(0)\ ,\ \ \widehat{u}_i^L \to 0 \text{ as } \widehat{x} \to \infty \end{cases}, \tag{5.6}$$

$$-\left(\widehat{v}_{i+2}^L\right)'' = \sum_{k=0}^{i} \left(\frac{a_{21}^{(k)}(0)}{k!} \mu^k \widehat{x}^k \widehat{u}_{i-k}^L + \frac{a_{22}^{(k)}(0)}{k!} \mu^k \widehat{x}^k \widehat{v}_{i-k}^L \right), \tag{5.7}$$

with

$$\widehat{v}_0^R = \widehat{v}_1^R = \widehat{v}_0^L = \widehat{v}_1^L = 0\ ,\ \ u_{-i} = 0\ \forall\, i > 0,$$

$$f_i = \begin{cases} f & \text{if } i = 0, \\ 0 & \text{otherwise} \end{cases}, \quad g_i = \begin{cases} g & \text{if } i = 0, \\ 0 & \text{otherwise} \end{cases}.$$

(The functions $\widehat{u}_i^R, \widehat{v}_i^R$ satisfy similar problems as (5.6) and (5.7), respectively.)

5.1 Analysis of the functions $\mathbf{U}_i$, $\widehat{\mathbf{U}}_i^L$, $\widehat{\mathbf{U}}_i^R$

5.1.1 Regularity of the functions $\mathbf{U}_i$, $\widehat{\mathbf{U}}_i^L$, $\widehat{\mathbf{U}}_i^R$

Theorem 5.2 *Let f, g, $\mathbf{A}$ satisfy (2.3), (2.7). Let $v_i, u_i, \widehat{u}_i^L, \widehat{v}_i^L, \widehat{u}_i^R, \widehat{v}_i^R$ be the solutions of (5.4)–(5.7), respectively. Let $G \subset \mathbb{C}$ be a complex neighborhood of $\overline{I}$ and set $\beta = \sqrt{a_{11}(0)} \in \mathbb{R}$. Then the functions $\widehat{u}_i^L$ and $\widehat{v}_i^L$, $\widehat{u}_i^R$ and $\widehat{v}_i^R$, are entire, and the functions u_i, v_i are analytic in a (fixed) neighborhood of I. Furthermore, there exist positive constants C, C', K, and $\gamma > 0$, independent of ε and μ, such that*

$$\widehat{v}_0^L = \widehat{v}_1^L = 0, \tag{5.8}$$

$$\left|\widehat{v}_{i+2}^L(z)\right| \leq CK^i \left(\mu + \frac{1}{i+1}\right)^i \frac{1}{i!} \left(iC' + |z|\right)^{2i} e^{-\beta \operatorname{Re}(z)}, \tag{5.9}$$

$$\left|\widehat{u}_i^L(z)\right| \leq CK^i \left(\mu + \frac{1}{i+1}\right)^i \frac{1}{i!} \left(iC' + |z|\right)^{2i} e^{-\beta \operatorname{Re}(z)}, \tag{5.10}$$

$$\|\mathbf{U}_i^{(n)}\|_{L^\infty(I)} \leq CK^i \left((i+n)\mu + 1\right)^i \gamma^n \max\{n, \mu^{-1}\}^n \qquad \forall n \in \mathbb{N}_0, \quad \forall i \in \mathbb{N}_0. \tag{5.11}$$

Furthermore, analogous estimates hold for $\widehat{u}_i^R$ and $\widehat{v}_i^R$ with β being now $\sqrt{a_{11}(1)}$.

Proof This is shown using induction in i, as was done in Theorem 4.3—for details, see [8, Theorem 5.5]. □

We conclude this section with a result that states that the boundary layer functions are in fact entire:

Lemma 5.3 *The functions $\widehat{\mathbf{U}}_i^L = (\widehat{u}_i, \widehat{v}_i)^T$ are entire functions, and there exist constants C, γ_1, γ_2, $\beta > 0$, independent of i, j, n, such that for all $\widehat{x} \geq 0$,*

$$|\widehat{u}_i^{(n)}(\widehat{x})| + |\widehat{v}_{i+2}^{(n)}(\widehat{x})| \leq Ce^{-\beta\widehat{x}}\gamma_1^i(i\mu+1)^i\gamma_2^n \qquad \forall n \in \mathbb{N}_0.$$

Proof Combine Theorem 5.2 with Cauchy's Integral Theorem for derivatives—for details, see [8, Lemma 5.6]. □

From Lemma 5.3, we get by a simple summation argument the following result, which expresses the fact that the contribution $\widehat{\mathbf{U}}_{BL}^M$ (and analogously,

$\widehat{\mathbf{V}}^M_{BL}$) in the decomposition (5.1) is of boundary layer character in the sense of Definition 2.1 if the expansion order is at most $O(1/\varepsilon)$:

Theorem 5.4 *Let f, g, and $\mathbf{A}$ satisfy (2.3) and (2.7). There exist constants C, γ, K, $b > 0$ such that, for $\gamma\{(M+1)\varepsilon + \frac{\varepsilon}{\mu}\} \le 1$, the function*

$$\widehat{\mathbf{U}}^M_{BL}(\widehat{x}) := \sum_{i=0}^{M} \left(\frac{\varepsilon}{\mu}\right)^i \widehat{\mathbf{U}}_i(\widehat{x}),$$

satisfies for $\widehat{x} > 0$,

$$\left|\frac{d^n}{d\widehat{x}^n}\widehat{\mathbf{U}}^M_{BL}(\widehat{x})\right| \le Ce^{-b\widehat{x}}K^n \qquad \forall n \in \mathbb{N}_0.$$

An analogous result holds for $\widehat{\mathbf{V}}^M_{BL} := \sum_{i=0}^{M}(\varepsilon/\mu)^i \widehat{\mathbf{U}}^R_i$.

5.2 Remainder estimates

We now turn to estimating the remainder, obtained by truncating the formal expansion (5.1). We write

$$\mathbf{U}(x) = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = \mathbf{W}_M(x) + \widehat{\mathbf{U}}^M_{BL}(\widehat{x}) + \widehat{\mathbf{V}}^M_{BL}(\widehat{x}) + \mathbf{R}_M(x), \tag{5.12}$$

where

$$\mathbf{W}_M(x) = \sum_{i=0}^{M} \left(\frac{\varepsilon}{\mu}\right)^i \begin{pmatrix} u_i(x) \\ v_i(x) \end{pmatrix}, \tag{5.13}$$

denotes the outer (smooth) expansion,

$$\widehat{\mathbf{U}}^M_{BL}(\widehat{x}) = \sum_{i=0}^{M} \left(\frac{\varepsilon}{\mu}\right)^i \begin{pmatrix} \widehat{u}_i^L(\widehat{x}) \\ \widehat{v}_i^L(\widehat{x}) \end{pmatrix}, \quad \widehat{\mathbf{V}}^M_{BL}(\widehat{x}) = \sum_{i=0}^{M} \left(\frac{\varepsilon}{\mu}\right)^i \begin{pmatrix} \widehat{u}_i^R(\widehat{x}^R) \\ \widehat{v}_i^R(\widehat{x}^R) \end{pmatrix}, \tag{5.14}$$

denote the left and right inner (boundary layer) expansions, respectively, and

$$\mathbf{R}_M(x) := \mathbf{U}(x) - \left(\mathbf{W}_M(x) + \widehat{\mathbf{U}}^M_{BL}(\widehat{x}) + \widehat{\mathbf{V}}^M_{BL}(\widehat{x}^R)\right) \tag{5.15}$$

denotes the remainder. We will estimate $L_{\varepsilon,\mu}\mathbf{W}_M - \mathbf{F}$, $L_{\varepsilon,\mu}\widehat{\mathbf{U}}^M_{BL}$, and $L_{\varepsilon,\mu}\widehat{\mathbf{V}}^M_{BL}$ separately. In that direction, we have the following theorem.

Theorem 5.5 *Let f, g, and $\mathbf{A}$ satisfy (2.3) and (2.7). Let $\mathbf{U}$ be the solution to the problem (2.1). Then there exist C, $\gamma > 0$, independent of μ, ε, and M, such that for $\mathbf{W}_M$ and $\widehat{\mathbf{U}}_M$ given by (5.13), (5.14), the following holds: If*

$$\left(\varepsilon(M+1) + \frac{\varepsilon}{\mu}\right)\gamma \le 1,$$

then

$$\left\| L_{\varepsilon,\mu}\left(\mathbf{U}-\mathbf{W}_M\right)\right\|_{L^\infty(I)} \le C\left[\left(\gamma\frac{\varepsilon}{\mu}\right)^{M+1} + (\gamma(M+1)\varepsilon)^{M+1}\right],$$

$$\left|L_{\varepsilon,\mu}\widehat{\mathbf{U}}_{BL}^M(\widehat{x})\right| \le Ce^{-\beta/2\widehat{x}}\begin{cases}(\varepsilon(M+1)+1/\mu)^{M-1} & \text{if } \widehat{x}\varepsilon\gamma \le 1\\ 1 & \text{if } \widehat{x}\varepsilon\gamma > 1\end{cases}.$$

Furthermore, the remainder $\mathbf{R}_M := \mathbf{U} - \left[\mathbf{W}_M + \widehat{\mathbf{U}}_{BL}^M + \widehat{\mathbf{V}}_{BL}^M\right]$ *satisfies the following estimate at the two endpoints of* I *for a constant* $b > 0$ *that is independent of* ε *and* μ*:*

$$\|\mathbf{R}_M\|_{L^\infty(\partial I)} \le Ce^{-b/\varepsilon},$$

Proof By and large, the proof follows from the asymptotic expansions. Again, the dichotomy $\varepsilon\widehat{x}\gamma \le 1$ and $\varepsilon\widehat{x}\gamma > 1$ results from the fact that the Taylor expansion of $\mathbf{A}$ converges only in a neighborhood of the expansion point $x = 0$. □

5.3 Proof of Theorem 5.1

The result of Theorem 5.1 now follows from combining Theorems 5.5 and 5.4. As in the proof of Theorem 4.1, we select $M \approx \lambda 1/\varepsilon$ for sufficiently small λ independent of ε and μ.

6 The second two scale case: Case IV

Recall that this occurs when $\mu/1$ is deemed small but ε/μ is *not* small. The main result is:

Theorem 6.1 *Assume that* f*,* g*, and* $\mathbf{A}$ *satisfy* (2.3)*,* (2.7)*. The solution* $\mathbf{U}$ *of* (2.1) *can be written as* $\mathbf{U} = \mathbf{W} + \widetilde{\mathbf{U}}_{BL} + \mathbf{R}$*, where* $\mathbf{W} \in \mathcal{A}(1, C_W, \gamma_W)$ *and* $\widetilde{\mathbf{U}}_{BL} \in \mathcal{BL}^2(\delta\mu, C_{BL}, \gamma_{BL})$*, for suitable constants* C_W*,* C_{BL}*,* γ_W*,* γ_{BL}*,* $\delta > 0$*, independent of* μ *and* ε*. Furthermore,* $\mathbf{R}$ *satisfies*

$$\|\mathbf{R}\|_{L^\infty(\partial I)} + \|L_{\varepsilon,\mu}\mathbf{R}\|_{L^2(I)} \le C(\mu/\varepsilon)^2 e^{-b/\mu},$$

for some constants C*,* $b > 0$*, independent of* μ *and* ε*. In particular,* $\|\mathbf{R}\|_{E,I} \le (\mu/\varepsilon)^2 e^{-b/\mu}$*.*

Proof See Section 6.4. □

In this case we recall the stretched variables $\widetilde{x} = x/\mu$ and $\widetilde{x}^R = (1-x)/\mu$, and make the formal ansatz

$$\mathbf{U} \sim \sum_{i=0}^{\infty} \mu^i \left[\mathbf{U}_i(x) + \widetilde{\mathbf{U}}_i^L(\widetilde{x}) + \widetilde{\mathbf{U}}_i^R(\widetilde{x}^R)\right], \tag{6.1}$$

where again

$$\mathbf{U}_i(x) = \begin{pmatrix} u_i \\ v_i \end{pmatrix}, \qquad \mathbf{U}_i^L(\widetilde{x}) = \begin{pmatrix} \widetilde{u}_i(\widetilde{x}) \\ \widetilde{v}_i(\widetilde{x}) \end{pmatrix},$$

and an analogous definition for $\widetilde{\mathbf{U}}^R$. We also recall that the differential operator $L_{\varepsilon,\mu}$ takes the form (4.9) on the $\widetilde{x}$-scale. The separation of the slow (x) and the fast variables ($\widetilde{x}$ and $\widetilde{x}^R$), leads to the following formal equations:

$$\sum_{i=0}^{\infty} \mu^i \left(-\mathbf{E}^{\varepsilon,\mu}\mathbf{U}_i'' + \mathbf{A}(x)\mathbf{U}_i\right) = \mathbf{F} = \begin{pmatrix} f \\ g \end{pmatrix}, \tag{6.2a}$$

$$\sum_{i=0}^{\infty} \mu^i \left(-\mu^{-2}\mathbf{E}^{\varepsilon,\mu}(\widetilde{\mathbf{U}}_i^L)'' + \sum_{k=0}^{\infty} \mu^k \widetilde{x}^k \mathbf{A}_k \widetilde{\mathbf{U}}_i^L\right) = 0, \tag{6.2b}$$

and an analogous system for $\widetilde{\mathbf{U}}^R$. Next, for ε appearing in (6.2) we write $\varepsilon = \mu\frac{\varepsilon}{\mu}$ and equate like powers of μ, to get with the matrix

$$\mathbf{E}^{\varepsilon/\mu,1} = \begin{pmatrix} \left(\frac{\varepsilon}{\mu}\right)^2 & 0 \\ 0 & 1 \end{pmatrix},$$

the following two recurrence relations:

$$-\mathbf{E}^{\varepsilon/\mu,1}\mathbf{U}_{i-2}'' + \mathbf{A}(x)\mathbf{U}_i = \mathbf{F}_i, \tag{6.3a}$$

$$-\mathbf{E}^{\varepsilon/\mu,1}(\widetilde{\mathbf{U}}_i^L)'' + \sum_{k=0}^{i} \widetilde{x}^{i-k}\mathbf{A}_{i-k}\widetilde{\mathbf{U}}_k^L = 0, \tag{6.3b}$$

where, as before, functions with negative index are assumed to be zero and $\mathbf{F}_i$ are defined by $\mathbf{F}_0 = (f, g)^T$ and $\mathbf{F}_i = 0$ for $i > 0$. The terms of the outer expansion, $\mathbf{U}_i$, are obtained immediately from (6.3a):

$$\mathbf{A}\mathbf{U}_i = \begin{pmatrix} f_i \\ g_i \end{pmatrix} + \begin{pmatrix} (\varepsilon/\mu)^2 u_{i-2}'' \\ v_{i-2}'' \end{pmatrix}. \tag{6.4}$$

The functions $\widetilde{\mathbf{U}}_i^L$ of the inner expansion are defined as the solutions of the following boundary value problems:

$$-\mathbf{E}^{\varepsilon/\mu,1}(\widetilde{\mathbf{U}}_i^L)'' + \mathbf{A}(0)\mathbf{U}_i^L = -\sum_{n=0}^{i-1} \widetilde{x}^{i-n}\widetilde{A}_{i-n}^L\widetilde{\mathbf{U}}_n^L, \tag{6.5a}$$

$$\widetilde{\mathbf{U}}_i^L(0) = -\mathbf{U}_i(0), \qquad \widetilde{\mathbf{U}}_i^L(\widetilde{x}) \to 0 \quad \text{as } \widetilde{x} \to \infty, \tag{6.5b}$$

and an analogous system for $\mathbf{U}_i^R$.

6.1 Properties of some solution operators

The functions $\mathbf{U}_i^L$ of the inner expansion are solutions to the elliptic *system* (6.5). In contrast to the previous arguments, which were based on estimates for *scalar* problems (for which strong tools such as maximum principles are readily available), we employ more general energy type arguments here to deal with the case of systems. For the reader's convenience, we give a few more details on our tools, namely, exponentially weighted spaces. On the half-line $(0, \infty)$, we define the norm

$$\|u\|_{0,\beta}^2 := \int_{x=0}^{\infty} e^{2\beta x} |u(x)|^2 \, dx, \tag{6.6}$$

with the obvious interpretation in case u is vector valued. The following lemma shows that elliptic systems of the relevant type (6.5) can be solved in a setting of exponentially weighted spaces:

Lemma 6.2 *Let $\nu \in (0, 1]$ and set $\mathbf{E} := \begin{pmatrix} \nu^2 & 0 \\ 0 & 1 \end{pmatrix}$. Let $\mathbf{B} \in \mathbb{R}^2$ be positive definite, i.e., $x^\top \mathbf{B} x \geq \beta_0^2 |x|^2$ for all $x \in \mathbb{R}^2$. Then the bilinear form*

$$a(\mathbf{U}, \mathbf{V}) = \int_0^\infty \mathbf{U}' \cdot \mathbf{E}\mathbf{V}' + \mathbf{U} \cdot \mathbf{B}\mathbf{V} \, dx,$$

satisfies for a constant $C > 0$ that depends solely on β_0, the following inf-sup condition for all $0 < \beta < \beta_0$: For any $\mathbf{U} \in H^1_\beta(0, \infty)$, there exists $\mathbf{V} \in H^1_{-\beta}(0, \infty)$ with $\mathbf{V} \neq 0$ such that

$$a(\mathbf{U}, \mathbf{V}) \geq C \frac{1}{\beta_0 - \beta} \|\mathbf{U}\|_{1,\beta} \|\mathbf{V}\|_{1,-\beta}.$$

Here, we define for $\alpha \in \mathbb{R}$ the space $H^1_\alpha(0, \infty) = \{u : ||u||_{1,\alpha} < \infty\}$ by the norm

$$\|\mathbf{U}\|_{1,\alpha}^2 := \int_{x=0}^{\infty} e^{2\alpha x} \left[\mathbf{U}' \cdot \mathbf{E}\mathbf{U}' + \mathbf{U} \cdot \mathbf{B}\mathbf{U}\right] dx.$$

Proof Given $\mathbf{U} \in H^1_\beta(0, \infty)$ we set $\mathbf{V} := e^{2\beta x}\mathbf{U}$. Then $\mathbf{V}'(x) = 2\beta e^{2\beta x}\mathbf{U}(x) + e^{2\beta x}\mathbf{U}'(x)$ and thus

$$\|\mathbf{V}\|_{1,-\beta}^2 \leq 4(1 + \frac{|\beta|}{\beta_0})^2 \|\mathbf{U}\|_{1,\beta}^2,$$

$$a(\mathbf{U}, \mathbf{V}) = \|\mathbf{U}\|_{1,\beta}^2 + 2\beta \int_{x=0}^{\infty} e^{2\beta x} \mathbf{U}' \cdot \mathbf{E}\mathbf{U} \, dx \geq \left(1 - \frac{|\beta|}{\beta_0}\right) \|\mathbf{U}\|_{1,\beta}^2.$$

The result then follows. □

Lemma 6.3 *Let $\mathbf{f}$ satisfy $\|\mathbf{f}\|_{0,\beta} < \infty$ for some $\beta \in [0, \beta_0)$, and let $\mathbf{g} \in \mathbb{R}^2$. Then the solution $\mathbf{U}$ of*

$$-\mathbf{E}\mathbf{U}'' + \mathbf{B}\mathbf{U} = \mathbf{f} \quad in\ (0, \infty), \qquad \mathbf{U}(0) = \mathbf{g},$$

satisfies for a $C > 0$ independent of β, the estimate

$$\|\mathbf{U}\|_{1,\beta} \leq C(\beta_0 - \beta)^{-1} \left[\|\mathbf{f}\|_{0,\beta} + |\mathbf{g}|\right].$$

Proof Let $\beta_1 > \beta_0$ and set $\mathbf{U}_0 = \mathbf{g}e^{-\beta_1 x}$. Then $\mathbf{U}_0$ satisfies the desired estimates. The remainder $\mathbf{U} - \mathbf{U}_0$ satisfies an inhomogeneous differential equation with homogeneous boundary conditions at $x = 0$. Hence, Lemma 6.2 is applicable and yields the desired result. □

6.2 Regularity of the functions $\mathbf{U}_i$, $\widetilde{\mathbf{U}}_i^L$, $\widetilde{\mathbf{U}}_i^R$

Lemma 6.4 *The function $\mathbf{U}_i$ defined by (6.4) are holomorphic in a neighborhood $G \subset \mathbb{C}$ of $\overline{I}$ and satisfy there, for some C, $\widetilde{K} > 0$,*

$$|\mathbf{U}_i(z)| \leq \widehat{C}\delta^{-i}\widehat{K}^i i^i \qquad \forall z \in G_\delta := \{z \in G \,|\, \operatorname{dist}(z, \partial G) > \delta\}, \qquad \forall i \geq 0.$$

Additionally, $\mathbf{U}_{2i+1} = 0$ for all $i \in \mathbb{N}_0$.

Proof We note that $\varepsilon/\mu \leq 1$. The arguments are then analogous to those of [5, Lemma 2]. The arguments are also structurally similar to the more complicated case studied in Lemma 4.2. □

We now turn to estimates for the inner expansion functions $\widetilde{\mathbf{U}}_i^L$.

Theorem 6.5 *Let f, g, $\mathbf{A}$ satisfy (2.3), (2.7). The functions $\widetilde{\mathbf{U}}_i^L$ defined by (6.5a), (6.5b) are entire functions and satisfy for all $\beta \in (0, \beta_0)$ (with $\beta_0 = \alpha$ given by (2.7))*

$$\|\widetilde{\mathbf{U}}_i^L\|_{1,\beta} \leq \widetilde{C}K^i(\beta_0 - \beta)^{-(2i+1)}i^i \qquad \forall i \in \mathbb{N}_0. \tag{6.7}$$

Proof We note that Lemma 6.3 can be applied with $\mathbf{B} = \mathbf{A}(0)$ and $\mathbf{E} = \mathbf{E}^{\nu,1}$ for $\nu = \varepsilon/\mu$. The case $i = 0$ follows from Lemmas 6.3 and 6.4. For $i \geq 1$, we proceed by induction. □

We next refine the argument to include bounds on all derivatives of $\widetilde{\mathbf{U}}_i$:

Theorem 6.6 *Let f, g, $\mathbf{A}$ satisfy (2.3), (2.7). Write $\beta_0 = \alpha$ and $\nu = \frac{\varepsilon}{\mu}$. Then there exist C_U, K_1, $K_2 > 0$ independent of $\beta \in (0, \beta_0)$, m, i, and ε, μ such that*

$$\left\| \frac{d^m}{d\widetilde{x}^m}\widetilde{\mathbf{U}}_i^L \right\|_{0,\beta} \leq C_U(\beta_0 - \beta)^{-(2i+1+m)}(i+m)^i K_1^i K_2^m \nu^{-m}.$$

Proof The cases $m = 0$ and $m = 1$ are covered by the above Lemma (note: $\|\mathbf{E}^{-1}\| \leq \nu^{-2}$). The remaining cases are obtained, as usual, by differentiating the equation satisfied by $\widetilde{\mathbf{U}}_i^L$ and then proceed by induction on m—see [8, Theorem 6.7] for details. □

6.3 Remainder estimates

As before, the formal expansion (6.1) is truncated after M terms to yield the decomposition

$$\mathbf{U} = \sum_{i=0}^{M} \mu^i \mathbf{U}_i(x) + \sum_{i=0}^{M} \mu^i \widetilde{\mathbf{U}}_i^L(\widetilde{x}) + \sum_{i=0}^{M} \mu^i \widetilde{\mathbf{U}}_i^R(\widetilde{x}^R) + \mathbf{R}^M$$
$$=: \mathbf{W}_M + \widetilde{\mathbf{U}}_{BL}^M + \widetilde{\mathbf{V}}_{BL}^M + \mathbf{R}_M. \tag{6.8}$$

We have for the boundary layer part $\widetilde{\mathbf{U}}_{BL}^M$ (and correspondingly for $\widetilde{\mathbf{V}}_{BL}^M$):

Theorem 6.7 *Let f, g, $\mathbf{A}$ satisfy (2.3), (2.7). Fix $\beta < \beta_0 = \alpha$ and set $\nu = \varepsilon/\mu$. There exist constants C, γ, $K > 0$ such that under the assumption $\mu(M+1)\gamma \leq 1$, there holds for $\widetilde{\mathbf{U}}_{BL}^M$ of (6.8)*

$$\left\| \frac{d^m}{d\widetilde{x}^m} \widetilde{\mathbf{U}}_{BL}^M \right\|_{0,\beta} \leq CK^m \nu^{-m} \qquad \forall m \in \mathbb{N}_0.$$

Proof From Theorem 6.6, we see that for all $m \in \mathbb{N}_0$,

$$\|(\widetilde{\mathbf{U}}_{BL}^M)^{(m)}\|_{0,\beta} \leq C(\beta_0-\beta)^{-1-2m} \sum_{i=0}^{M} \mu^i (i+m)^i (\beta_0-\beta)^{-2i} K_1^i K_2^m \nu^{-m}$$
$$\leq C(\beta_0-\beta)^{-1-2m} K_2^m \nu^{-m} \sum_{i=0}^{M} (2(\beta_0-\beta)^{-2} K_1 \mu i)^i + (2K_1\mu m)^i$$
$$\leq C(\beta_0-\beta)^{-1-2m} K_2^m \nu^{-m} \sum_{i=0}^{M} (2K_1 \mu M)^i \left((\beta_0-\beta)^{-2i} + \left(\frac{m}{M}\right)^i \right)$$
$$\leq C\widetilde{K}_2^m \nu^{-m}$$

for an appropriate $\widetilde{K}_2$ (depending on $\beta_0 - \beta$), if we assume that μM is sufficiently small. The key observation for this fact is to note that for $m > M$ we have $n^{1/n} \to 1$ for $n \to \infty$ and

$$\sum_{i=0}^{M} (m/M)^i \leq (M+1)(m/M)^M = m\frac{M+1}{M}(m/M)^{M-1}$$
$$\leq m\frac{M+1}{M}\left(\frac{m}{M-1}\right)^{((M-1)/m)m}.$$

□

It is worth pointing out that our convention $\widetilde{x} = x/\mu$ implies that (for M satisfying the assumption of Theorem 6.7), we have that $\widetilde{\mathbf{U}}_{BL}^M$, viewed as a function of x, is an element of $\mathcal{BL}^2(\delta\mu, C, \gamma\nu)$ for suitable δ, C, $\gamma > 0$ independent of ε, μ (see Definition 2.1). From the Sobolev embedding theorem

$\|v\|^2_{L^\infty(J)} \leq C\|v\|_{L^2(J)}\|v\|_{H^1(J)}$ for any interval J of unit length, we then obtain—by writing $(0,\infty) = \cup_{j=0}^{\infty}(j, j+1)$ that the function $\widetilde{\mathbf{U}}^M_{BL}$, again viewed as a function of x, is an element of $\mathcal{BL}^\infty(\delta\mu, C\sqrt{\nu}, \gamma\nu)$ for suitable $\delta, C, \gamma > 0$. This is the assertion of Case IV of Theorem 2.2.

For the residual, we have:

Theorem 6.8 *Let f, g, $\mathbf{A}$ satisfy* (2.3), (2.7). *There exists $\gamma > 0$ independent of ε, μ such that under the assumption*

$$\mu(M+1)\gamma \leq 1,$$

there holds for $\mathbf{W}_M$ and $\widetilde{\mathbf{U}}^M_{BL}$ given by (6.8),

$$\|\mathbf{F} - L_{\varepsilon,\mu}\mathbf{W}_M\|_{L^\infty(I)} \leq C(\mu(M+1)\gamma)^{M+2},$$
$$\|L_{\varepsilon,\mu}\widetilde{\mathbf{U}}^M_{BL}\|_{L^2(I)} \leq C\sqrt{\mu}\left[(\mu(M+1)\gamma)^{M+1} + \left(\frac{\mu}{\varepsilon}\right)^2 e^{-b/\mu}\right],$$

where C, $b > 0$ are independent of ε and μ. The remainder $\mathbf{R}_M$ of (6.8) *satisfies the following estimates at the endpoints:*

$$\|\mathbf{R}_M\|_{L^\infty(\partial I)} \leq C\left(\frac{\mu}{\varepsilon}\right)^{1/2} e^{-b/\mu},$$

for C, $b > 0$ independent of μ and ε.

Proof Similar to the proof of Theorem 5.5—see [8, Theorems 6.9, 6.11, 6.12] for details. □

6.4 Proof of Theorem 6.1

The proof of Theorem 6.1 now follows by making the choice $M \approx \lambda/\mu$ for sufficiently small λ (independent of μ and ε) and combining Theorems 6.8 and 6.7 and using the stability result (2.9).

7 Numerical experiment

As discussed in [9, 18], the regularity assertions of the present paper allow one to show that the *hp*-version of the finite element method (*hp*-FEM) can lead to robust exponential convergence for the problem class (2.1). Key to this convergence result are specially designed meshes that are capable to resolve the boundary layers at the endpoints $x = 0$ and $x = 1$. In an *hp*-FEM setting, the minimal mesh that is able to resolve boundary layers on the $O(\varepsilon)$- and $O(\mu)$-scale is the *Spectral Boundary Layer mesh*:

Definition 7.1 (Spectral Boundary Layer mesh) For a transition parameter $\kappa > 0$, a polynomial degree $p \in \mathbb{N}$ and $0 < \varepsilon \leq \mu \leq 1$, the *spectral boundary* *layer mesh* Δ for the interval $(0,1)$ is determined by the following mesh points:

$$\Delta = \{0, \kappa p\varepsilon, \kappa p\mu, 1-\kappa p\mu, 1-\kappa p\varepsilon, 1\} \qquad \text{if } \kappa p\mu < \frac{1}{2}$$

$$\Delta = \{0, \kappa p\varepsilon, 1-\kappa p\varepsilon, 1\} \qquad \text{if } \kappa p\varepsilon < \frac{1}{2} \le \kappa p\mu$$

$$\Delta = \{0, 1\} \qquad \text{if } \kappa p\varepsilon \ge \frac{1}{2}$$

For the *hp*-FEM approximation of the solution of (2.1), we take as the finite element space $V_N \subset \left[H_0^1(I)\right]^2$ the conforming space of piecewise polynomials of degree p (in each component) on the Spectral Boundary Layer mesh. The FEM approximation $\mathbf{U}_{\text{FEM}} \in V_N$ is the best approximation to the exact solution $\mathbf{U}$ in the energy norm $\|\cdot\|_E$. It is shown in [9, Theorem 3.3] that for κ sufficiently small (depending on the analytic data $\mathbf{A}$, $\mathbf{f}$) we have the *a priori* estimate

$$\|\mathbf{U} - \mathbf{U}_{\text{FEM}}\|_E \le Ce^{-\sigma\kappa p}, \tag{7.1}$$

where the constants $C, \sigma > 0$ are independent of p, ε, and μ. In our numerical studies, we select $\kappa = 1$, and use the matrix $\mathbf{A}$ given below. We consider two different cases given by the following right-hand sides $\mathbf{F}^1$ and $\mathbf{F}^2$:

$$\mathbf{A} = \begin{pmatrix} 2(x+1)^2 & -\left(1+x^2\right) \\ -2\cos(\pi x/4) & 2.2e^{1-x} \end{pmatrix}, \quad \mathbf{F}^1(x) = \begin{pmatrix} 2e^x \\ 10x+1 \end{pmatrix},$$

$$\mathbf{F}^2(x) = \tfrac{1}{x+1/8}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

In our computations, we select $\varepsilon = \mu^{3/2}$, thus leading to solutions with three scales. The numerical results are presented in Fig. 1, where the relative energy norm error (in percent) is plotted versus the polynomial degree p. In the parameter regime studied here, the meshes always consist of five elements. The two plots in Fig. 1 contain an additional line, which is the relative error (in percent) of the L^2-projection of the leading order smooth part $\mathbf{U}_{00}$ onto the polynomials of degree p on I. Both plots show robust exponential convergence of the FEM as predicted by (7.1). The right-hand side $\mathbf{F}^1$ shows, in the regime of polynomial degrees p and values μ shown here, a better bound in μ, namely,

$$\|\mathbf{U} - \mathbf{U}_{\text{FEM}}\|_E \le C\sqrt{\mu}e^{-bp}, \tag{7.2}$$

for some $C, b > 0$ independent of μ. This behavior can be understood qualitatively as follows (and could be made rigorous by studying the proof of [9, Theorem 3.3]): By Theorem 2.2 the exact solution $\mathbf{U}$ can be decomposed as

$$\mathbf{U} = \mathbf{W} + \widetilde{\mathbf{U}}_{BL} + \widehat{\mathbf{U}}_{BL} + \mathbf{R}.$$

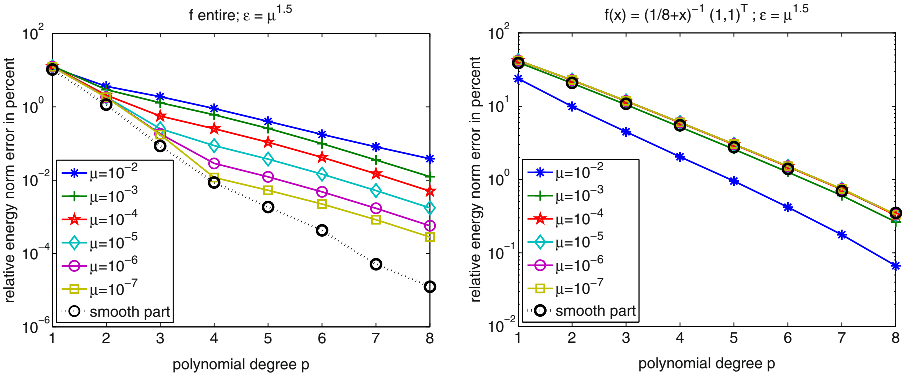

Fig. 1 Convergence in the energy norm for right-hand side $\mathbf{F}^1$ (*left*) and $\mathbf{F}^2$ (*right*)

When approximating the terms $\mathbf{W}$, $\widetilde{\mathbf{U}}_{BL}$, and $\widehat{\mathbf{U}}_{BL}$ on the Spectral Boundary Layer mesh separately, one expects to be able to find piecewise polynomials $\mathbf{W}_N$, $\widetilde{\mathbf{U}}_N$, $\widehat{\mathbf{U}}_N$ and constants $C, b, b_1, b_2 > 0$ such that

$$\|\mathbf{W} - \mathbf{W}_N\|_E \leq Ce^{-bp},$$
$$\|\widetilde{\mathbf{U}}_{BL} - \widetilde{\mathbf{U}}_N\|_E \leq C\sqrt{\mu}e^{-b_1\kappa p},$$
$$\|\widehat{\mathbf{U}}_{BL} - \widehat{\mathbf{U}}_N\|_E \leq C\sqrt{\varepsilon}e^{-b_2\kappa p}.$$

Hence, ignoring the error introduced by disregarding the remainder $\mathbf{R}$, we expect an upper bound for the FEM convergence of

$$O(e^{-bp} + \sqrt{\mu}e^{-b_1\kappa p} + \sqrt{\varepsilon}e^{-b_2\kappa p}). \tag{7.3}$$

Since $\varepsilon = \mu^{3/2}$, the term $\sqrt{\varepsilon}e^{-b_2\kappa p}$ may be dropped when trying to understand the convergence behavior in its dependence on μ. We note the following special case: if $b \gg b_1\kappa$, then in a (large) regime of value of the polynomial degree p and values μ, the total error is dominated by the second contribution $\sqrt{\mu}e^{-b_1\kappa p}$. This is indeed visible in the left plot in Fig. 1. To illustrate that indeed $b > b_1\kappa$, we include in Fig. 1 the L^2-projection error for the leading order smooth part $\mathbf{U}_{00}$, which is a good approximation to $\mathbf{W}$.... We observe that for these values of p and μ, the total error is dominated by the approximation of the boundary layer part on the $O(\mu)$-scale. It is worth noting that the situation in which the approximation of the layer part dominates the convergence in spite of the presence of a smooth part is rather specific to the present p-version approach and is unlikely to occur in an h-version setting of classical Shishkin meshes: It is a fortunate combination of a large domain of analyticity of the smooth part $\mathbf{W}$ with the rapid exponential convergence of poynomial approximation. To underline this observation, we now study the case of the right-hand side $\mathbf{F}^2$, where the domain of analyticity of the smooth part is rather small. Correspondingly, we expect the total error to be dominated by the approximation of the smooth part. This is visible in the right plot in Fig. 1 for μ in the range $\mu = 10^{-3}$—10^{-7}, where the approximation of the smooth part (represented in the plot by $\mathbf{U}_{00}$) practically fully accounts the total approximation error. We mention two further references with numerical studies for the right-hand side $\mathbf{F}^1$: [9] studies different combinations of ε and μ and includes a discussion of maximum norm convergence; [18] focusses on the case $\varepsilon = \mu$.

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