hp Finite Element Methods for Fourth Order Singularly Perturbed Boundary Value Problems

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Abstract. We consider fourth order singularly perturbed boundary value problems (BVPs) in one-dimension and the approximation of their solution by the *hp* version of the Finite Element Method (FEM). If the given problem's boundary conditions are suitable for writing the BVP as a second order system, then we construct an *hp* FEM on the so-called *Spectral Boundary Layer Mesh* that gives a robust approximation that converges exponentially in the energy norm, provided the data of the problem is analytic. We also consider the case when the BVP is not written as a second order system and the approximation belongs to a finite dimensional subspace of the Sobolev space H^2. For this case we construct suitable C^1-conforming hierarchical basis functions for the approximation and we again illustrate that the *hp* FEM on the *Spectral Boundary Layer Mesh* yields a robust approximation that converges exponentially. A numerical example that validates the theory is also presented.

1 Introduction: The Model Problems

Singularly Perturbed Problems (SPPs) arise in numerous applications from science and engineering, such as electrical networks, vibration problems, and in the theory of hydrodynamic stability [2]. In such problems, the highest derivative in the differential equation is multiplied by a very small positive parameter. This causes the solution to contain *boundary layers*, which are rapidly varying solution components that have support in a narrow neighbourhood of the boundary of the domain. Their numerical approximation must be carefully constructed in order for their effects to be accurately captured [8]. In the context of the FEM, this requires the use of *graded meshes* which include refinement near the boundary layer region that depends on the singular perturbation parameter. Examples of such meshes include the Bakhvalov [1] and Shishkin [10] meshes, which are used with finite differences and the *h* version of the FEM, as well as the *Spectral Boundary Element Mesh* [3] which is used with the *p* and *hp* versions of the FEM.

Most SPPs that have been studied in the literature concern second order differential operators; notable exceptions are the works [2,4,6,9]. In this paper we

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consider fourth order elliptic boundary value problems (BVP) with two different types of boundary conditions: one that is suitable for writing the BVP as a second order system and one that is not. The model problem we study here is a simplified version of the well-known Orr-Sommerfeld equation from hydrodynamics (cf. [2]), and reads: Find $u(x)$ such that

$$\varepsilon^2 u^{(4)}(x) - \alpha(x)u''(x) + \beta(x)u(x) = f(x) \text{ in } I = (0,1)\,, \tag{1}$$

where $\alpha \geq 0, \beta \geq 0$ and f are given (sufficiently smooth) functions and $\varepsilon \in (0,1]$ is a given parameter (that can approach 0). The notation u'' means the second derivative of u with respect to x and $u^{(n)}, n \in \mathbb{N}$, denotes the n^{th} derivative of u with respect to x. Equation (1) is supplemented with one of the following two types of boundary conditions (which for simplicity are chosen as homogeneous):

$$u(0) = u'(0) = u(1) = u'(1) = 0, \tag{2}$$

or

$$u(0) = u''(0) = u(1) = u''(1) = 0. \tag{3}$$

As we will see in the next section, the BVP (1), (3) can be written as a second order system and its approximation will follow the work of [3]. For the BVP (1), (2) we will construct a C^1 approximation from a finite dimensional subspace of H^2, using a hierarchical basis from [7]. The results presented here summarize the analysis found in [3] and [5].

Throughout the paper we will utilize the notation $H^k(I)$ to mean the usual Sobolev space of functions defined on I, whose $0, 1, ..., k$ (generalized) derivatives belong to $L^2(I)$, with associated norm and seminorm $\|\cdot\|_{k,I}\,, |\cdot|_{k,I}$, respectively; the Lebesgue spaces $L^p(I), 1 \leq p \leq \infty$, are defined in the usual way and $\langle\cdot,\cdot\rangle_I$ denotes the usual $L^2(I)$ inner-product. We will also use the space

$$H^1_0(I) = \left\{u \in H^1(I) : u|_{\partial I} = 0\right\},$$

where ∂I denotes the boundary of I. Finally, the letters c, C (with or without subscripts) will denote generic positive constants independent of the solution and any discretization parameters.

2 Second Order Systems

We first consider the problem (1), (3), which can be written as the following second order system: Find $\mathbf{U}(x) = [u_1(x), u_2(x)]^T (= [u''(x), u(x)]^T)$ such that

$$-\mathbf{E}\mathbf{U}''(x) + \mathbf{A}(x)\mathbf{U}(\mathbf{x}) = \mathbf{F}(x) \text{ in } I,\ \mathbf{U}(a) = \mathbf{U}(b) = \mathbf{0}, \tag{4}$$

where

$$\mathbf{E} = \begin{bmatrix} \varepsilon^2 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{A}(x) = \begin{bmatrix} \alpha(x) & -\beta(x) \\ 1 & 0 \end{bmatrix}, \mathbf{F}(x) = \begin{bmatrix} -f(x) \\ 0 \end{bmatrix}. \tag{5}$$

For the remainder of the paper, we assume that the data of our problem are analytic and that there exist positive constants $C_\alpha, C_\beta, C_f, \gamma_\alpha, \gamma_\beta, \gamma_f$ independent of ε such that $\forall\ n = 0, 1, 2, ...$

$$\left\|\alpha^{(n)}\right\|_{L^\infty(I)} \le C_\alpha \gamma_\alpha^n n! \ , \ \left\|\beta^{(n)}\right\|_{L^\infty(I)} \le C_\beta \gamma_\beta^n n! \ , \ \left\|f^{(n)}\right\|_{L^\infty(I)} \le C_f \gamma_f^n n! \ . \quad (6)$$

Moreover, we assume that the matrix valued function $\mathbf{A}(x)$ in (5) is pointwise positive definite (but not necessarily symmetric), i.e. for some fixed $\delta > 0$

$$\boldsymbol{\xi}^T \mathbf{A} \boldsymbol{\xi} \ge \delta^2 \boldsymbol{\xi}^T \boldsymbol{\xi} \quad \forall\ \boldsymbol{\xi} \in \mathbb{R}^2 \ , \quad \forall\ x \in \bar{I}. \quad (7)$$

The variational formulation of (4) reads: Find $\mathbf{U} := (u_1, u_2) \in \left[H_0^1(I)\right]^2$ such that

$$\mathcal{B}(\mathbf{U}, \mathbf{V}) = \mathcal{F}(\mathbf{V}) \ \ \forall\ \mathbf{V} = (v_1, v_2) \in \left[H_0^1(I)\right]^2 , \quad (8)$$

where

$$\mathcal{B}(\mathbf{U}, \mathbf{V}) = \varepsilon^2 \left\langle u_1', v_1' \right\rangle_I + \left\langle u_2', v_2' \right\rangle_I + \left\langle \alpha u_1 - \beta u_2, v_1 \right\rangle_I + \left\langle u_1, v_2 \right\rangle_I , \quad (9)$$

$$\mathcal{F}(\mathbf{V}) = \left\langle -f, v_2 \right\rangle_I . \quad (10)$$

It follows that the bilinear form $\mathcal{B}(\cdot, \cdot)$ given by (9), is coercive with respect to the *energy norm*

$$\|\mathbf{U}\|_{E,I}^2 \equiv \|(u_1, u_2)\|_{E,I}^2 := \varepsilon^2 |u_1|_{1,I}^2 + |u_2|_{1,I}^2 + \delta^2 \left(\|u_1\|_{0,I}^2 + \|u_2\|_{0,I}^2 \right), \quad (11)$$

i.e.,

$$\mathcal{B}(\mathbf{V}, \mathbf{V}) \ge \|\mathbf{V}\|_{E,I}^2 \quad \forall\ \mathbf{V} \in \left[H_0^1(I)\right]^2 . \quad (12)$$

This fact, along with the continuity of $\mathcal{B}(\cdot, \cdot)$ and $\mathcal{F}(\cdot)$ imply the unique solvability of (8).

The discrete version of (8) reads: Find $\mathbf{U}^N := (u_1^N, u_2^N) \in [S_N]^2 \subset \left[H_0^1(I)\right]^2$ such that

$$\mathcal{B}(\mathbf{U}^N, \mathbf{V}) = \mathcal{F}(\mathbf{V}) \ \ \forall\ \mathbf{V} = (v_1, v_2) \in [S_N]^2 , \quad (13)$$

where S_N is a finite dimensional subspace of $H_0^1(I)$, to be defined shortly. The unique solvability of (13) follows from (7), and by the well-known Galerkin orthogonality, we have

$$\left\|\mathbf{U} - \mathbf{U}^N\right\|_{E,I}^2 \le C \inf_{\mathbf{V} \in [S_N]^2} \|\mathbf{U} - \mathbf{V}\|_{E,I}^2 \quad \forall\ \mathbf{V} \in [S_N]^2 . \quad (14)$$

Before we define the space S_N we comment on the regularity of the solution to (8) and in particular we quote a relevant result from [3] that shows that the solution can be decomposed into an outer (smooth) part, a boundary layer part and a remainder that is exponentially (in ε) small.

Theorem 1. *[3] Assume (6) and (7) hold. Then there exist constants* $C, \gamma, q, \nu > 0$ *independent of* $\varepsilon \in (0,1]$ *such that the following assertions are true for the solution of (8):*

(I) $\left\|\mathbf{U}^{(n)}\right\|_{L^\infty(I)} \leq C\varepsilon^{-1/2}\gamma^n \max\left\{n, \varepsilon^{-1}\right\}^n \;\forall\; n = 0,1,2,...$

(II) $\mathbf{U}$ *can be written as* $\mathbf{U} = \mathbf{W} + \mathbf{U}_{BL} + \mathbf{R}$, *with*

$$\left\|\mathbf{W}^{(n)}\right\|_{L^\infty(I)} \leq C\gamma^n n^n \;\forall\; n = 0,1,2,...,$$

$$\left|\mathbf{U}_{BL}^{(n)}(x)\right| \leq C\gamma^n \left(\nu\varepsilon\right)^{-n} e^{-dist(x,\partial I)/\nu\varepsilon} \;\forall\; n = 0,1,2,...,$$

$$\|\mathbf{R}\|_{L^\infty(\partial I)} + \|\mathbf{R}\|_{E,I} \leq Ce^{-q/\varepsilon}.$$

Moreover, the second component u_2^{BL} *of* $\mathbf{U}_{BL}$*, satisfies the stronger estimate*

$$\left|(u_2^{BL})^{(n)}(x)\right| \leq C\gamma^n\varepsilon^2 \left(\nu\varepsilon\right)^{-n} e^{-dist(x,\partial I)/\nu\varepsilon} \;\forall n = 0,1,2,...$$

Practically speaking, the above theorem states that for ε relatively large, the solution of (8) is analytic if the data α, β, f are analytic. If, on the other hand ε is small, then the solution may be decomposed into the three aforementioned parts and estimates on the derivatives of each part are given. This information is the key ingredient for the proof of (exponential) convergence of the proposed method.

We now define the space S_N: Let $\Delta = \{0 = x_0 < x_1 < ... < x_M = 1\}$ be an arbitrary partition of $I = (0,1)$ and set $I_j = (x_{j-1}, x_j), h_j = x_j - x_{j-1}, j = 1,...,M$. We also define the master (or standard) element $I_{ST} = (-1,1)$ and note that it can be mapped onto the j^{th} element I_j by the linear mapping

$$x = Q_j(t) = \frac{1}{2}(1-t)x_{j-1} + \frac{1}{2}(1+t)x_j.$$

With $\Pi_p(I_{ST})$ denoting the space of polynomials of degree $\leq p$ on I_{ST}, we define the (finite dimensional) subspace

$$S^p(\Delta) = \left\{\mathbf{V} \in \left[H_0^1(I)\right]^2 : \mathbf{V} \circ Q_j^{-1} \in \left(\Pi_{p_j}(I_{ST})\right)^2, j = 1,...,M\right\},$$

where $\circ$ denotes composition of functions. Then, we set

$$S_N \equiv S_0^p\left(\Delta\right) = S^p\left(\Delta\right) \cap \left[H_0^1(I)\right]^2. \tag{15}$$

We restrict our attention here to constant polynomial degree p for all elements, i.e. $p_j = p \;\forall\; j$, but clearly more general settings with variable polynomial degrees are possible. The following *Spectral Boundary Layer mesh* is essentially the minimal mesh that yields exponential convergence.

Definition 1. *(Spectral Boundary Layer mesh) For* $\kappa > 0, p \in \mathbb{N}$ *and* $0 < \varepsilon \leq 1$, *define the spaces* $S\left(\kappa, p\right)$ *of piecewise polynomials by*

$$S\left(\kappa, p\right) := \begin{cases} S_0^p\left(\Delta\right); \Delta = \{0,1\} & \text{if } \kappa p\varepsilon \geq \frac{1}{2} \\ S_0^p\left(\Delta\right); \Delta = \{0, \kappa p\varepsilon, 1-\kappa p\varepsilon, 1\} & \text{if } \kappa p\varepsilon < \frac{1}{2} \end{cases}$$

The parameter κ is user specific and depends on the problem under consideration as well as the length scales of the boundary layers – we refer to [8] for a more detailed discussion of this issue and we note that in practice the value $\kappa = 1$ yields satisfactory results for most problems. We also note that the method we are considering is not a true *hp* FEM since the location and not the number of the elements changes; a more correct characterization would be a *p* version FEM on a moving mesh, but in order to be consistent with the bibliography we utilize the term *hp* FEM for our method. Obviously, additional refinement and/or using a true *hp* version would yield better results but at the cost of using more degrees of freedom – see [11] for a numerical comparison.

The main result is the following:

Theorem 2. *[3] Let* $\mathbf{U}$ *be the solution to (8) and let* $\mathbf{U}^N \in S(\kappa, p)$ *be the solution of (13) with* $S(\kappa, p)$ *given by Definition 1. Then there exist constants* $\kappa_0, C, \sigma > 0$ *depending only on the data* α, β, f*, such that for any* $0 < \kappa \leq \kappa_0$

$$\left\| \mathbf{U} - \mathbf{U}^N \right\|_{E,I} \leq Ce^{-\sigma\kappa p}.$$

3 A C^1 Approximation

In this section we consider the BVP (1), (2) whose variational formulation reads: Find $u \in H_0^2(I) := \left\{ u \in H^2(I) : u(0) = u'(0) = u(1) = u'(1) = 0 \right\}$, such that

$$\overline{\mathcal{B}}(u, v) = \overline{\mathcal{F}}(v) \ \forall \ v \in H_0^2(I), \tag{16}$$

where

$$\overline{\mathcal{B}}(u, v) = \int_0^1 \left\{ \varepsilon^2 u''(x)v''(x) + \alpha(x)u'(x)v'(x) + \beta(x)u(x)v(x) \right\} dx, \tag{17}$$

$$\overline{\mathcal{F}}(v) = \int_0^1 f(x)v(x)dx. \tag{18}$$

We continue to assume analyticity of the input data, i.e. (6), and coercivity of the bilinear form $\overline{\mathcal{B}}(\cdot, \cdot)$ holds in the *energy norm*

$$|||u|||_{E,I}^2 := \overline{\mathcal{B}}(u, u). \tag{19}$$

Existence and uniqueness follow from the Lax-Milgram lemma as usual. The discrete version of (16) reads: Find $u_N \in \overline{S}_N \subset H_0^2(I)$ such that

$$\overline{\mathcal{B}}(u_N, v) = \overline{\mathcal{F}}(v) \ \forall \ v \in \overline{S}_N, \tag{20}$$

and we have

$$|||u - u_N|||_{E,I} \leq C \inf_{v \in \overline{S}_N} |||u - v|||_{E,I} \quad \forall \ v \in \overline{S}_N.$$

In order to define the space $\overline{S}_N$, we introduce, for $t \in I_{ST}$, the four *nodal* basis functions (cf. [7])

$$N_1(t) = \frac{1}{4}(1-t)^2(2+t), N_2(t) = \frac{1}{4}(1+t)^2(2-t),$$
$$N_3(t) = \frac{1}{4}(1-t)(2+t)^2, N_4(t) = \frac{1}{4}(1+t)(2-t)^2,$$

as well as the $p-3$ *internal* basis functions

$$N_i(t) = \frac{1}{\sqrt{2(i-5)}}\left(\frac{1}{2i-7}P_{i-5}(t) - \frac{2(i-5)}{(2i-7)(2i-3)}P_{i-3}(t) + \frac{1}{2i-3}P_{i-1}(t)\right),$$

for $i \geq 5$, where $P_i(t)$ is the Legendre polynomial of degree i. The (first two) nodal basis functions are equal to 1 at one endpoint of I_{ST} and 0 at the other. The internal basis functions are 0 at both endpoints of I_{ST}. There holds

$$\Pi_p(I_{ST}) = \operatorname{span}\{N_1(t), ..., N_{p+1}(t)\},$$

and, with Δ, Q_j as in the previous section, we define the space

$$S^p(\Delta) := \left\{w \in H_0^2(I) : w \circ Q_j^{-1} \in \Pi_{p_j}(I_{ST}), j = 1, ..., m\right\}.$$

Hence, the subspace $\overline{S}_N \subset H_0^2(I)$ is chosen as

$$\overline{S}_N \equiv S_0^p(\Delta) = S^p(\Delta) \cap H_0^2(I). \tag{21}$$

The following proposition gives bounds on the n^{th} derivative of the solution.

Proposition 1. *[5] Assume (6) and (7) hold. Then there exist constants* $C, \gamma, \nu, q > 0$ *independent of* $\varepsilon \in (0,1]$ *such that the following assertions are true for the solution of (16):*

(I) $\left\|u^{(n)}\right\|_{L^\infty(I)} \leq C\gamma^n \max\left\{n^n, \varepsilon^{1-n}\right\} \; \forall\, n = 0,1,2,...$

(II) u *can be written as* $u = w + u_{BL} + r$, *with*

$$\left\|w^{(n)}\right\|_{L^\infty(I)} \leq C\gamma^n n^n \; \forall\, n = 0,1,2,...,$$
$$\left|u_{BL}^{(n)}(x)\right| \leq C\gamma^n \varepsilon^2 (\nu\varepsilon)^{-n} e^{-dist(x,\partial I)/\nu\varepsilon} \; \forall\, n = 0,1,2,...,$$
$$\|r\|_{L^\infty(\partial I)} + |||r|||_{E,I} \leq Ce^{-q/\varepsilon}.$$

Using the above result we can prove the following.

Proposition 2. *[5] Let* u *be the solution to (16) and let* $u_N \in S(\kappa, p)$ *be the solution of (20) with* $S(\kappa, p)$ *given by Definition 1. Then there exist constants* $\kappa_0, C, \sigma > 0$ *depending only on the data* α, β, f*, such that for any* $0 < \kappa \leq \kappa_0$

$$|||u - u_N|||_{E,I} \leq Ce^{-\sigma\kappa p}.$$

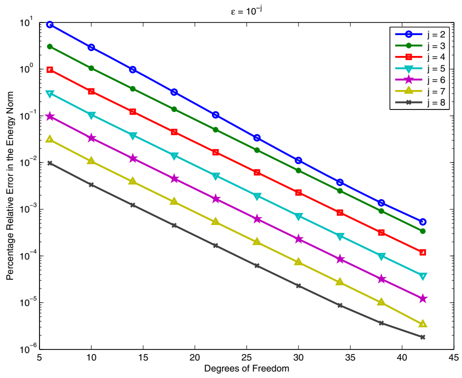

Fig. 1. Energy norm convergence for the *hp* version

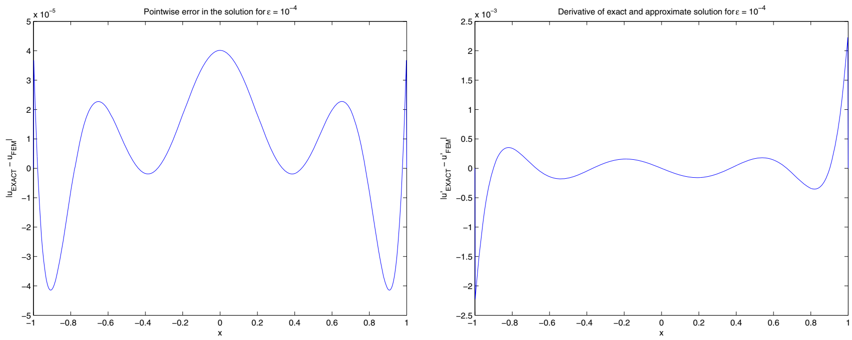

Fig. 2. Pointwise error in the approximation (left) and in the derivatives (right), for $\varepsilon = 10^{-4}$

4 Numerical Results

In this section we present the results of numerical computations for the problem studied in the previous section; ample numerical results pertaining to the problem studied in Section 2 may be found in [3] and [11]. We consider the BVP (1), (2) when the data is $\alpha = \beta = f = 1$; an exact solution is available, hence our reported results are reliable. Figure 1 shows the percentage relative error in the energy norm,

$$Error := 100 \times \frac{|||u - u_N|||_{E,I}}{|||u|||_{E,I}}, \tag{22}$$

versus the number of degrees of freedom, N, in a semi-log scale. As the figure shows, the error curves are straight, something that verifies the exponential convergence of the proposed method. Moreover, as $\varepsilon \to 0$ the method not only does not deteriorate, but it actually performs better. This suggests that in the error estimate of Proposition 2, there is a positive power of ε present, something that is due to the fact that the problem has constant coefficients and the right hand side belongs to the subspace (see [8] for more details on this for second order scalar problems).

Figure 2 shows the pointwise error between the exact and *hp* approximation (computed with $p = 8$) as well as their derivatives, for $\varepsilon = 10^{-4}$. The high accuracy of the computed solution is readily visible from these figures.

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