The *hp* finite element approximation of a weakly coupled system of two singularly perturbed reaction-diffusion equations

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Abstract

We consider the approximation of a weakly coupled system of two singularly perturbed reaction-diffusion equations, with the finite element method. The first differential equation contains a small parameter multiplying the highest derivative, while the second does not. As a result, the first component of the solution to the system will contain boundary layers and our goal is to construct and analyze an *hp* finite element scheme which converges uniformly with respect to the singular perturbation parameter. In particular, our scheme includes elements of size \(O(\varepsilon p)\) near the boundary, where \(\varepsilon\) is the singular perturbation parameter and \(p\) is the degree of the approximating polynomials. We show that under the assumption of analytic input data, the method yields exponential rates of convergence, independently of \(\varepsilon\), as \(p \to \infty\). Numerical computations supporting the theory are also presented.

1 Introduction

The numerical solution of (scalar) singularly perturbed boundary value problems has received a lot of attention during the last two decades. It is well known that the main difficulty in these problems
is the presence of boundary layers in the solution, whose accurate approximation, independently of
the singular perturbation parameter(s), has been the main focus of numerous research endeavors
(see, e.g., the books [10], [11], [14] and the references therein). In the context of the Finite Element
Method (FEM), the robust approximation of boundary layers requires either the use of the h version
on non-uniform meshes (such as the Shishkin [17] or Bakhvalov [1] mesh), or the use of the high
order p and hp versions on specially designed (variable) meshes [16]. In both cases, the a-priori
knowledge of the position of the layers is taken into account, and mesh-degree combinations can be
chosen for which uniform error estimates can be established (cf. [3], [9], [16]).

In recent years researchers have turned their attention to systems of singularly perturbed problems,
which have two (or more) overlapping boundary layers, such as the one considered below: Find \( \overline{u} \)
such that

\[
L \overline{u} := \begin{bmatrix} -\varepsilon^2 \frac{d^2}{dx^2} & 0 \\ 0 & -\mu^2 \frac{d^2}{dx^2} \end{bmatrix} \overline{u} + A \overline{u} = \overline{f} \quad \text{in} \quad \Omega = (0,1) \tag{1}
\]

where \( 0 < \varepsilon \leq \mu \leq 1 \),

\[
A = \begin{bmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{bmatrix}, \quad \overline{f}(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} \tag{2}
\]

along with the boundary conditions on \( \partial \Omega \)

\[
\overline{u}(0) = \overline{\gamma}_0, \quad \overline{u}(1) = \overline{\gamma}_1. \tag{3}
\]

The data \( \varepsilon, \mu, A, \overline{f}, \overline{\gamma}_0 \) and \( \overline{\gamma}_1 \) are given and the unknown solution is \( \overline{u}(x) = [u_1(x), u_2(x)]^T \). Under certain assumptions on the data (see eqs. (6), (7) below) there exists a unique solution to
(1), (3) whose overall regularity depends on the data in the usual way (cf. [6]).

The presence of \( \varepsilon \) and \( \mu \) in (1) causes the solution \( \overline{u} \) to have boundary layers near the endpoints
of \( \Omega \), which, in general, overlap and interact. Problems of this type arise in the modelling of
turbulence in water waves [19], as well as in the finite element approximation of shells, where the
singular perturbation parameters are related to the thickness of the shell [12]. Matthews et al.
[7, 8], studied the above problem for the cases \( 0 < \varepsilon = \mu << 1 \) and \( 0 < \varepsilon \ll \mu = 1 \), obtaining an
approximation using finite differences which converged independently of \( \varepsilon \) and \( \mu \). The more general
case of \( 0 < \varepsilon \leq \mu \leq 1 \) was studied by Madden and Stynes [6] and by Linß and Madden [4, 5] in
the context of finite differences, and by Linß and Madden [3] in the context of the \( h \) version of the
FEM with piecewise linear basis functions. In all the works mentioned, estimates were obtained
showing that the approximation converged (at the expected rate) independently of \( \varepsilon \) and \( \mu \). In terms
of higher order \( p \) and \( hp \) FEMs, the analysis for case \( 0 < \varepsilon = \mu < 1 \) appears in [21] and for the case
\( 0 < \varepsilon < \mu < 1 \) in [22]; see also [20] for a numerical study comparing all versions of the FEM.

In the present article we study the case \( 0 < \varepsilon \leq \mu = 1 \) by considering the weakly coupled system
of two singularly perturbed reaction-diffusion equations shown in equation (4) below, and we focus
on its approximation by the \( hp \) version of the FEM. As mentioned above, the analogous problem
with two equal singular perturbation parameters was studied in [21], and the one with two distinct
singular perturbation parameters was studied in [22]. In the present case we want to find \( \overline{u} = [u_1(x), u_2(x)]^T \) such that

\[
L \overline{u} := \begin{bmatrix} -\varepsilon^2 \frac{d^2}{dx^2} & 0 \\ 0 & -\mu^2 \frac{d^2}{dx^2} \end{bmatrix} \overline{u} + A \overline{u} = \overline{f} \quad \text{in} \quad \Omega = (0,1), \tag{4}
\]

\[
\overline{u}(0) = 0, \quad \overline{u}(1) = 0. \tag{5}
\]
where $0 < \varepsilon \leq 1$, and $A, \overrightarrow{f}$ are given by (2). We will assume that
\begin{equation}
    a_{12}(x) \leq 0, \quad a_{21}(x) \leq 0 \quad \forall \ x \in \overline{\Omega},
\end{equation}
and
\begin{equation}
    \min_{\overline{\Omega}} \{ a_{11}(x) + a_{12}(x), a_{21}(x) + a_{22}(x) \} \geq \alpha^2 > 0
\end{equation}
for some constant $\alpha \in \mathbb{R}$, in order to guarantee that $A$ is invertible and $\|A^{-1}\|$ is bounded [2]. Note that it follows from (6)–(7) that $a_{11}(x) \geq \alpha^2 > 0 \quad \forall \ x \in \overline{\Omega}$.

The presence of $\varepsilon$ in (4) causes the first component of the solution $\overrightarrow{u}$ to have boundary layers near the endpoints of $\Omega$. This is illustrated in figure 1 below, which shows the two components of $\overrightarrow{u}$ for the case
\[
A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad \overrightarrow{f}(x) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \varepsilon = 10^{-2}.
\]

![Figure 1: The exact solution for $\varepsilon = 10^{-2}$.

The rest of the paper is organized as follows: In Section 2 we discuss the properties of the solution to (4)–(5). In Section 3 we present the finite element formulation and the design of the $hp$ scheme we will be considering, along with our main result of exponential convergence. Section 4 contains the results of some numerical computations illustrating the exponential convergence of the method, and finally, in Section 5 we summarize our conclusions.

In what follows, the space of squared integrable functions on an interval $\Omega \subset \mathbb{R}$ will be denoted by $L^2(\Omega)$, with associated inner product
\[
(u, v)_{\Omega} := \int_{\Omega} uv.
\]
We will also utilize the usual Sobolev space notation $H^k(\Omega)$ to denote the space of functions on $\Omega$ with $0, 1, 2, ..., k$ generalized derivatives in $L^2(\Omega)$, equipped with norm and seminorm $\|\cdot\|_{k, \Omega}$ and $|\cdot|_{k, \Omega}$, respectively. For vector functions $\overrightarrow{u} = [u_1(x), u_2(x)]^T$, we will write
\[
\|\overrightarrow{u}\|_{k, \Omega}^2 = \|u_1\|_{k, \Omega}^2 + \|u_2\|_{k, \Omega}^2.
\]
We will also use the space 

\[ H^1_0(\Omega) = \{ u \in H^1(\Omega) : u|_{\partial \Omega} = 0 \}, \]

where \( \partial \Omega \) denotes the boundary of \( \Omega \). Finally, the letter \( C \) will be used to denote a generic positive constant, independent of any discretization or singular perturbation parameters and possibly having different values in each occurrence.

## 2 The Model Problem and its Regularity

We consider the model problem (4)–(5), described in the previous section, and we assume that the functions \( a_{ij}(x) \) and \( f_i(x) \) are analytic on \( \overline{\Omega} \) and that there exist constants \( C_f, \gamma_f, C_a, \gamma_a > 0 \) such that

\[
\left\| f_i^{(n)} \right\|_{\infty, \Omega} \leq C_f \gamma_f^n \quad \forall n \in \mathbb{N}_0, \ i = 1, 2, \tag{8}
\]

\[
\left\| a_{ij}^{(n)} \right\|_{\infty, \Omega} \leq C_a \gamma_a^n \quad \forall n \in \mathbb{N}_0, \ i, j = 1, 2. \tag{9}
\]

As usual, we cast the problem (4)–(5) into an equivalent weak formulation, which reads: Find \( \vec{u} \in \mathbb{R}^2 \) such that

\[
B (\vec{u}, \vec{v}) = F (\vec{v}) \quad \forall \vec{v} \in \mathbb{R}^2, \tag{10}
\]

where

\[
B (\vec{w}, \vec{v}) = \varepsilon^2 (u'_1, v'_1) + (u'_2, v'_2) + (a_{11}u_1 + a_{12}u_2, v_1) + (a_{21}u_1 + a_{22}u_2, v_2), \tag{11}
\]

\[
F (\vec{v}) = (f_1, v_1) + (f_2, v_2). \tag{12}
\]

From (7), we get that for any \( x \in \overline{\Omega} \),

\[
\overline{\xi}^T A \overline{\xi} \geq \alpha^2 \overline{\xi}^T \overline{\xi} \quad \forall \overline{\xi} \in \mathbb{R}^2, \tag{13}
\]

and it follows that the bilinear form \( B (\cdot, \cdot) \) is coercive with respect to the energy norm

\[
\| \vec{u} \|^2_{E, \Omega} := \varepsilon^2 |u'_{1,1, \Omega}|^2 + |u'_{2,1, \Omega}|^2 + \alpha^2 \left( \| u_1 \|^2_{0, \Omega} + \| u_2 \|^2_{0, \Omega} \right), \tag{14}
\]

i.e.,

\[
B (\vec{w}, \vec{v}) \geq \| \vec{w} \|^2_{E, \Omega} \quad \forall \vec{w} \in \mathbb{R}^2. \tag{15}
\]

This, along with the continuity of \( B (\cdot, \cdot) \) and \( F (\cdot) \), imply the unique solvability of (10). We also have the following a priori estimate

\[
\| \vec{u} \|_{E, \Omega} \leq \frac{1}{\alpha} \| \vec{f} \|_{0, \Omega}. \tag{16}
\]

For the discretization, we choose a finite dimensional subspace \( S_N \) of \( H^1_0(\Omega) \) and solve the problem: Find \( \vec{u}_N \in \mathbb{R}^2 \) such that

\[
B (\vec{u}_N, \vec{v}) = F (\vec{v}) \quad \forall \vec{v} \in \mathbb{R}^2. \tag{17}
\]
The unique solvability of the discrete problem (17) follows from (13) and (15), and by the well-known orthogonality relation we have
\[
\| \overrightarrow{u} - \overrightarrow{u}_N \|_{E, \Omega} \leq \inf_{\overrightarrow{v} \in [S_N]^2} \| \overrightarrow{u} - \overrightarrow{v} \|_E.
\]  

We now present results on the regularity of the solution to (4)–(5). These follow the analysis found in [9] for the analogous scalar problem, and are based on the results from [22] for a singularly perturbed system with two singular perturbation parameters. Note that by the analyticity of \(a_{ij}\) and \(f_i\), we have that \(u_i\) are analytic. Moreover, we have the following theorem from [22].

**Theorem 1.** Let \(\overrightarrow{u}\) be the solution to (4)–(5) with \(0 < \varepsilon \leq 1\). Then there exist constants \(C\) and \(K > 0\), independent of \(\varepsilon\), such that
\[
\| \overrightarrow{u}^{(n)} \|_{0, \Omega} \leq CK^n \max\{n, \varepsilon^{-1}\}^n \quad \forall \ n \in \mathbb{N}_0.
\]  

We will now obtain a decomposition for the solution \(\overrightarrow{u}\) into a smooth (asymptotic) part, two boundary layer parts and a remainder as follows:
\[
\overrightarrow{u} = \overrightarrow{w} + A^- \overrightarrow{u}^- + A^+ \overrightarrow{u}^+ + \overrightarrow{r}.
\]  

This decomposition is obtained by inserting the formal ansatz
\[
u_k(x) \sim \sum_{j=0}^{\infty} \varepsilon^j \overrightarrow{u}_k^j(x), \ k = 1, 2
\]  

into the differential equation (4), and equating like powers of \(\varepsilon\), so that we can define the smooth part \(\overrightarrow{w}\) as
\[
\overrightarrow{w}(x) := \sum_{j=0}^{M} \varepsilon^j \overrightarrow{w}^j
\]  

where the terms \(\overrightarrow{w}^j = [u_1^j(x), u_2^j(x)]^T\) are defined as follows:

- If \(j\) is odd, \(\overrightarrow{w}^j = \overrightarrow{0}\).
- For \(j = 0\), \(u_2^0\) is the (unique) solution to the boundary value problem
\[
-(u_2^0)'' + \overrightarrow{w}_2^0 = \overrightarrow{T} \text{ in } \Omega, \ u_2^0(0) = u_2^0(1) = 0,
\]  

\[
\overrightarrow{T} = f_2 - \frac{a_{21}}{a_{11}} f_1, \overrightarrow{w} = \frac{a_{22}a_{11} - a_{12}a_{21}}{a_{11}} > 0 \forall x \in \overrightarrow{\Omega},
\]  

and
\[
u_1^0 = \frac{f_1 - a_{12}a_{21}^0}{a_{11}}.
\]
For \( j = 2, 4, 6, \ldots \), \( u_j^2 \) is the (unique) solution to the boundary value problem

\[
-u_j^2'' + \pi_j u_j^2 = -\frac{a_{j1}}{a_{11}} \left( u_j^{j-2} \right)'' \quad \text{in } \Omega, \quad u_j^2(0) = u_j^2(1) = 0,
\]

and

\[
u_j^2 = \frac{\left( u_j^{j-2} \right)'' - a_{12} u_j^2}{a_{11}}.
\]

**Remark 1.** Note that \( \mathbf{f} \) and \( \mathbf{a} \) in (24) are defined in terms of analytic functions satisfying (8), (9). It is straightforward to show that \( \mathbf{f} \) and \( \mathbf{a} \) are also analytic and that there exist constants \( C_\mathbf{f}, \gamma_\mathbf{f}, C_\mathbf{a}, \gamma_\mathbf{a} > 0 \) such that

\[
\left| \mathbf{f}^{(n)} \right|_{\infty, \Omega} \leq C_\mathbf{f} \gamma_\mathbf{f}^n n! \quad \text{and} \quad \left| \mathbf{a}^{(n)} \right|_{\infty, \Omega} \leq C_\mathbf{a} \gamma_\mathbf{a}^n n! \quad \forall n \in \mathbb{N}_0.
\]

Next, we note that by (22), (23) and (26) we have

\[
\overline{u}(0) := \sum_{j=0}^{M} \varepsilon^j \overline{u}^j(0) = \sum_{j=0}^{M} \varepsilon^j \begin{bmatrix} u_j^1(0) \\ 0 \end{bmatrix}
\]

and

\[
\overline{u}(1) := \sum_{j=0}^{M} \varepsilon^j \overline{u}^j(1) = \sum_{j=0}^{M} \varepsilon^j \begin{bmatrix} u_j^1(1) \\ 0 \end{bmatrix}.
\]

A calculation shows that

\[
L(\overline{u} - \overline{w}) = \begin{bmatrix} \varepsilon^{M+2} (u_1^M)'' \\ 0 \end{bmatrix},
\]

hence, as \( \varepsilon \to 0 \), \( \overline{w}(x) \) defined by (22) satisfies the differential equation, but not the boundary conditions. To correct this we introduce boundary layer functions \( \overline{u}^- \) and \( \overline{u}^+ \) by

\[
L(\overline{u}^-) = 0 \quad \text{in } \Omega, \quad L(\overline{u}^+) = 0 \quad \text{in } \Omega,
\]

\[
\overline{u}^-(0) = (1, 1)^T, \quad \overline{u}^+(0) = \overline{0}, \quad \overline{u}^-(1) = \overline{0}, \quad \overline{u}^+(1) = (1, 1)^T.
\]

Moreover, we define \( \overline{r} \) by

\[
L(\overline{r}) = \begin{bmatrix} \varepsilon^{M+2} (u_1^M)'' \\ 0 \end{bmatrix},
\]

\[
\overline{r}(0) = \overline{r}(1) = \overline{0}.
\]

Finally, in order to satisfy the boundary conditions (5) we must choose \( A^- \) and \( A^+ \) in (20) appropriately so that

\[
\overline{u}(0) \equiv \overline{w}(0) + A^- \overline{u}^-(0) + A^+ \overline{u}^+(0) + \overline{r}(0) = 0
\]

and

\[
\overline{u}(1) \equiv \overline{w}(1) + A^- \overline{u}^-(1) + A^+ \overline{u}^+(1) + \overline{r}(1) = 0.
\]

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Using (29), (30), (32) and (33), we see that we must set
\[
A^- = \begin{bmatrix}
-w_1(0) & 0 \\
0 & 0
\end{bmatrix},
\]
and
\[
A^+ = \begin{bmatrix}
-w_1(1) & 0 \\
0 & 0
\end{bmatrix}
\]
in (20). As a result, the asymptotic expansion for \( \overrightarrow{u}(x) \) becomes
\[
\overrightarrow{u}(x) = \overrightarrow{w}(x) + \begin{bmatrix}
-w_1(0)u_1^-(x) \\
0
\end{bmatrix} + \begin{bmatrix}
-w_1(1)u_1^+(x) \\
0
\end{bmatrix} + \overrightarrow{r}(x). \quad (34)
\]

We will now analyze the behavior of the terms in the decomposition of \( \overrightarrow{u} \).

**Lemma 2.** Let \( \overrightarrow{u}^j \) be defined as in (23)–(27). Then there exist constants \( C, K_1, K_2 > 0 \) independent of \( \varepsilon \) and \( n \) such that for \( \ell = 1, 2 \) and any \( j = 0, 2, 4, \ldots \)
\[
\left\| u^j_\ell^{(n)} \right\|_{0, \Omega} \leq CK_1^n K_2^j j! n! \quad \forall \ n \in \mathbb{N}_0.
\]

**Proof.** First we have the following estimates, which follow from (23)–(27) and standard stability estimates of elliptic boundary value problems (cf. [18]):
\[
\left\| u_2^0 \right\|_{\infty, \Omega} \leq C \left\| f \right\|_{\infty, \Omega},
\]
\[
\left\| u_2^0 \right\|_{\infty, \Omega} \leq C \frac{1}{\alpha^2} \left( \left\| f_1 \right\|_{\infty, \Omega} + \left\| a_{12} \right\|_{\infty, \Omega} \left\| f \right\|_{\infty, \Omega} \right),
\]
\[
\left\| u_2^{j+2} \right\|_{\infty, \Omega} \leq C \frac{1}{\alpha^2} \left\| a_{21} \right\|_{\infty, \Omega} \left\| u_1^j \right\|_{\infty, \Omega}^{(n)} \quad , \ j = 0, 2, 4, \ldots
\]
\[
\left\| u_2^{j+2} \right\|_{\infty, \Omega} \leq C \frac{1}{\alpha^4} \left( \alpha^2 + \left\| a_{12} \right\|_{\infty, \Omega} \left\| a_{21} \right\|_{\infty, \Omega} \right) \left\| u_1^j \right\|_{\infty, \Omega}^{(n)} \quad , \ j = 0, 2, 4, \ldots
\]
with the constants \( C > 0 \) independent of \( u_\ell^j \).

The proof follows that of Lemma 2 in [9] and is included for completeness. Given the assumption of analytic input data, there exists a complex neighborhood \( G \subset \mathbb{C} \) of \( \Omega \) such that \( f_i, a_{ij}, f \) and \( \pi \) are analytic on \( G \). Now, for \( 0 < \delta \leq 1 \) define the sets
\[
G_\delta := \{ z \in G : \text{dist}(z, \partial G) \geq \delta \}.
\]
We will establish the following (stronger) result which will then enable us to prove the lemma: There exist constants \( K, C > 0 \) such that for \( 0 < \delta \leq 1 \)
\[
\left\| u_\ell^j \right\|_{\infty, \Omega, G_\delta} \leq C\delta^{-j} K^j j! \left\| \overrightarrow{u}^0 \right\|_{\infty, G} \quad , \ \ell = 1, 2.
\]
(39)
Choose \( K^2 > \max \left\{ \frac{2e}{\alpha^2} \left( \alpha^2 + \left\| a_{12} \right\|_{\infty, G} \left\| a_{21} \right\|_{\infty, G} \right), \frac{2e}{\alpha^2} \left\| a_{21} \right\|_{\infty, G} \right\} \). Clearly (39) is true for \( j = 0 \).
We proceed by induction on \( j \), so we assume that (39) holds for a given \( j \geq 0 \) and establish it for
Choosing \( K \) in (39), let \( c \) terms of the data. Hence, letting \( (38) \), where the integration path is chosen as the circle of radius \( \kappa \delta \) about \( x \), to obtain
\[
\left\| u_1^{j+2} \right\|_{\infty,G_\delta} \leq \frac{C}{\alpha^4} \left( \alpha^2 + \| a_{12} \|_{\infty,\Omega} \| a_{21} \|_{\infty,\Omega} \right) \frac{2 \pi \kappa \delta}{2 \pi (\kappa \delta)^3} \left\| u_1^j \right\|_{\infty,G_{(1-\kappa)\delta}}.
\]

Using the induction hypothesis, we have
\[
\left\| u_1^{j+2} \right\|_{\infty,G_\delta} \leq \frac{2C}{\alpha^4} \left( \alpha^2 + \| a_{12} \|_{\infty,\Omega} \| a_{21} \|_{\infty,\Omega} \right) (\kappa \delta)^{-2} j! K^j ((1 - \kappa) \delta)^{-j} \left\| \overline{w}^0 \right\|_{\infty,G} \leq \delta^{-(j+2)} K^{j+2} (j + 2)! \left\| \overline{w}^0 \right\|_{\infty,G} \left[ \frac{2C}{\alpha^4} \frac{K^2 \kappa^2 (1 - \kappa)^j (j + 1)(j + 2)}{\alpha^4 K^2 \kappa^2 (1 - \kappa)^j (j + 1)(j + 2)} \right].
\]

Choosing \( \kappa = 1/(j + 2) \) we have
\[
\frac{1}{\kappa^2(1-\kappa)^j(j+1)(j+2)} = (j + 2)^{j+1} \frac{1}{(j + 1)^{j+1}} = \left( 1 + \frac{1}{j + 1} \right)^{j+1} \leq e \forall j \in \mathbb{N}_0,
\]

hence by the choice of \( K \) the expression in brackets above is bounded by 1 and we have
\[
\left\| u_1^{j+2} \right\|_{\infty,G_\delta} \leq C \delta^{-(j+2)} K^{j+2} (j + 2)! \left\| \overline{w}^0 \right\|_{\infty,G}, \quad \ell = 1.
\] (40)

Repeating the argument for \( u_2^{j+2} \), and using (37) while keeping in mind the choice of \( K \), gives (40) for \( \ell = 2 \).

Having established (39), let \( r = \min \{ \text{dist}(z, \partial G) \} \) and \( \delta = \min \{ 1, r/2 \} \). Then for any \( z \in \Omega \), the disk with radius \( \delta \) and center \( z \) is in \( G_\delta \subset G \). For any \( x \) on the circle with radius \( \delta \) and center \( z \) we have \( \left\| u_\ell^j (x) \right\| \leq \left\| u_\ell^j \right\|_{\infty,G_\delta} \). So by Cauchy’s Integral Theorem and (39) we have for any \( z \in \Omega \), and \( \ell = 1, 2 \),
\[
\left\| \left( \frac{d^n}{dz^n} \right) u_\ell^j (z) \right\| \leq \frac{n!}{\delta^n} \left\| u_\ell^j \right\|_{\infty,G_\delta} \leq \frac{n!}{\delta^n} \delta^{-j} K^j j! \left\| \overline{w}^0 \right\|_{\infty,G} \leq \frac{C}{\delta} \left( \frac{K}{\delta} \right)^j j! n! \left\| \overline{w}^0 \right\|_{\infty,\Omega}.
\]

Recall that \( K \) and \( \delta \) depend only on the data, and that by (35), (36) we can bound \( \left\| \overline{w}^0 \right\|_{\infty,\Omega} \) in terms of the data. Hence, letting \( K_1 = 1/\delta, K_2 = K/\delta \) we obtain the desired result.

The next theorem bounds the derivatives of \( \overline{w} \), independently of \( \varepsilon \).

**Theorem 3.** There exist constants \( C, K_1, K_2 \in \mathbb{R}^+ \) depending only on \( \overline{f} \) and \( A \) such that if \( \varepsilon K_2 M < 1, \overline{w}(x) \) given by (22), satisfies
\[
\left\| \overline{w}^{(n)} \right\|_{0,\Omega} \leq C K_2^n n! \forall n \in \mathbb{N}_0.
\] (41)
Proof. From (22) and Lemma 2 with $\overline{K}_2 = K_2$, we have
\[
\left\| \overrightarrow{u}^{(n)} \right\|_{0,\Omega} \leq \sum_{j=0}^{M} \varepsilon^j \left\| \overrightarrow{u}^j \right\|_{0,\Omega} \leq C \sum_{j=0}^{M} \varepsilon^j K_1^n \overline{K}_2^n n! \leq CK_1^n n! \sum_{j=0}^{M} \varepsilon^j \overline{K}_2^n j! \\
\leq CK_1^n n! \sum_{j=0}^{M} \varepsilon^j \overline{K}_2^n j! \leq CK_1^n n! \sum_{j=0}^{M} \varepsilon^j \overline{K}_2^n j! \leq CK_1^n n! \left( \varepsilon \overline{K}_2 M \right)^j.
\]
Since $\varepsilon \overline{K}_2 M < 1$ the above sum can be bounded by a converging geometric series, and we have the result. \( \square \)

Remark 2. By the previous theorem we have that $A^+$ and $A^-$ in the decomposition (20) (or (34)) for $\overrightarrow{u}$ are bounded independently of $\varepsilon$.

The following theorem gives bounds on the boundary layer part $\overrightarrow{u}^-$. Analogous bounds hold for $\overrightarrow{u}^+$.

**Theorem 4.** Let $\overrightarrow{u}^-$ be the solution of (32). Then there exist constants $C, K > 0$ independent of $\varepsilon$ and $n$ such that for any $x \in \overline{\Omega}$, $n \in \mathbb{N}_0$,
\[
\left| \left( \overrightarrow{u}^- \right)^{(n)} (x) \right| \leq CK^n e^{-\alpha x / \varepsilon} \max \{ n, \varepsilon^{-1} \} n.
\]

Proof. A more general version of this result has been established in [22] for a singularly perturbed system with two singular perturbation parameters; the result for the (simpler) system considered here follows from that. \( \square \)

The final theorem of this section gives bounds on the remainder $\overrightarrow{r}$ in terms of $\varepsilon$, the order $M$ of the asymptotic expansion and the input data.

**Theorem 5.** There exists constants $C, K_2 > 0$ depending only on the input data $A$ and $\overrightarrow{f}$ such that the remainder $\overrightarrow{r}$ defined by (33) satisfies
\[
\left\| \overrightarrow{r} \right\|_{E,\Omega} \leq C \varepsilon^2 (\varepsilon K_2 M)^M.
\]

Proof. Recall that $\overrightarrow{r}$ satisfies (33), hence we have the a-priori estimate
\[
\left\| \overrightarrow{r} \right\|_{E,\Omega} \leq \frac{1}{\alpha} \left\| \varepsilon^{M+2} \left( \overrightarrow{u}_1^M \right)^n \right\|_{0,\Omega}.
\]
By Lemma 2, we further obtain
\[
\left\| \overrightarrow{r} \right\|_{E,\Omega} \leq C \varepsilon^{M+2} K_1^2 K_2^M n! \leq CK_1^2 \varepsilon^2 (\varepsilon K_2 M)^M
\]
and we have the desired result. \( \square \)

Remark 3. Theorem 5 shows that the remainder is small provided $\varepsilon M$ is small. In the case when $\varepsilon M$ is large the asymptotic expansion loses its meaning.
3 The Finite Element Method

In this section we describe the specific choice of the subspace $S_N$, which will allow us to approximate the solution of (17) at an exponential rate.

Let $\Delta = \{0 = x_0 < x_1 < \ldots < x_M = 1\}$ be an arbitrary partition of $\Omega = (0, 1)$ and set

$$ I_j = (x_{j-1}, x_j), \quad h_j = x_j - x_{j-1}, \quad j = 1, \ldots, M. $$

Also, define the master (or standard) element $I_{ST} = (-1, 1)$, and note that it can be mapped onto the $j^{th}$ element $I_j$ by the linear mapping

$$ x = Q_j(t) = \frac{1}{2} (1 - t) x_{j-1} + \frac{1}{2} (1 + t) x_j. $$

With $\Pi_p(I_{ST})$ the space of polynomials of degree $\leq p$ on $I_{ST}$, we define our finite dimensional subspaces as

$$ S_N \equiv S^p(\Delta) = \{ u \in H^1_0(\Omega) : u(Q_j(t)) \in \Pi_p(I_{ST}), j = 1, \ldots, M \} $$

and

$$ S^p_0(\Delta) := [S^p(\Delta) \cap H^1_0(\Omega)]^2, \quad (44) $$

where $\overrightarrow{p} = (p_1, \ldots, p_M)$ is the vector of polynomial degrees assigned to the elements.

The following approximation result from [15] will be the main tool for the analysis of the method.

**Theorem 6.** For any $u \in C^\infty(I_{ST})$ there exists $I_p u \in \Pi_p(I_{ST})$ such that

$$ u(\pm 1) = I_p u(\pm 1), \quad (45) $$

$$ \| u - I_p u \|_{0, I_{ST}}^2 \leq \frac{1}{p^2} \frac{(p - s)!}{(p + s)!} \| u^{(s+1)} \|_{0, I_{ST}}^2, \quad \forall s = 0, 1, \ldots, p, \quad (46) $$

$$ \| (u - I_p u) \|_{0, I_{ST}}^2 \leq \frac{1}{p^2} \frac{(p - s)!}{(p + s)!} \| u^{(s+1)} \|_{0, I_{ST}}^2, \quad \forall s = 0, 1, \ldots, p. \quad (47) $$

The definition below describes the mesh used for the method: If we are in the asymptotic range of $p$, i.e. $p \geq 1/\varepsilon$, then a single element suffices since $p$ will be sufficiently large to give us exponential convergence without any refinement. If we are in the pre-asymptotic range, i.e. $p < 1/\varepsilon$, then the mesh consists of three elements as described below. We should point out that this is the minimal mesh-degree combination for attaining exponential convergence; obviously, refining within each element will retain the convergence rate but would require more degrees of freedom – one such example is the so-called geometrically graded mesh discussed in [9] for the scalar problem.

**Definition 7.** For $\kappa > 0$, $p \in \mathbb{N}$ and $0 < \varepsilon \leq 1$, define the spaces $S(\kappa, p)$ of piecewise polynomials by

$$ S(\kappa, p) := \begin{cases} S^p_0(\Delta); \quad \Delta = \{0, 1\} & \text{if } \kappa \varepsilon \geq \frac{1}{2} \\ S^p_0(\Delta); \quad \Delta = \{0, \varepsilon, 1 - \varepsilon, 1\} & \text{if } \kappa \varepsilon < \frac{1}{2} \end{cases} $$

In both cases above the polynomial degree is uniformly $p$ on all elements.
Before we state the main theorem of the paper, we cite a useful computation whose proof is straightforward (see also [22]).

**Lemma 8.** Let \( p \in \mathbb{N}, \lambda \in (0, 1) \). Then

\[
\frac{(p - \lambda p)!}{(p + \lambda p)!} \leq \left[ \frac{(1 - \lambda)^{(1-\lambda)}}{(1 + \lambda)^{(1+\lambda)}} \right]^p p^{-2\lambda p} e^{2\lambda p + 1}.
\]

We now present our main result.

**Theorem 9.** Let \( \mathbf{f} \) and \( A \) be composed of functions that are analytic on \( \overline{\Omega} \) and satisfy the conditions in (6)–(9). Let \( \mathbf{u} = [u_1, u_2]^T \) be the solution to (4)–(5). Then there exist constants \( \kappa, C, \beta > 0 \) depending only on \( \mathbf{f} \) and \( A \) such that there exists \( \mathcal{I}_p \mathbf{u} = [\mathcal{I}_p u_1, \mathcal{I}_p u_2]^T \in \mathcal{S}(\kappa, p) \) with \( \mathcal{I}_p \mathbf{u} = \mathbf{u} \) on \( \partial \Omega \) and

\[
\| \mathbf{u} - \mathcal{I}_p \mathbf{u} \|^2_{E, \Omega} \leq C p^3 e^{-\beta p}.
\]

**Proof.** The proof follows that of Theorem 11 from [22], and it is included for completeness.

**Case 1.** \( \kappa \varepsilon > \frac{1}{2} \), i.e. \( p \geq \frac{1}{2 \kappa \varepsilon} \) (asymptotic case), \( \Delta = \{0, 1\} \)

From Theorem 1 we have

\[
\left\| \nabla^{(n)} \mathbf{u} \right\|^2_{0, \Omega} \leq C K^{2n} \max\{n, \varepsilon^{-1}\}^{2n} \quad \forall n \in \mathbb{N}_0,
\]

and by Theorem 6 there exists \( \mathcal{I}_p \mathbf{u} \in \mathcal{S}(\kappa, p) \) such that \( \mathbf{u} = \mathcal{I}_p \mathbf{u} \) on \( \partial \Omega \) and for any \( s = 0, ..., p \)

\[
\| (\mathbf{u} - \mathcal{I}_p \mathbf{u})' \|^2_{0, \Omega} \leq \frac{(p - s)!}{(p + s)!} \| \nabla^{(s+1)} \mathbf{u} \|^2_{0, \Omega} \leq \frac{(p - s)!}{(p + s)!} C K^{2(s+1)} \max\{s + 1, \varepsilon^{-1}\}^{2(s+1)}.
\]

Choose \( s = \lambda p \), for some \( \lambda \in (0, 1] \). Then, since \( p \geq \frac{1}{2\kappa \varepsilon} \), we have

\[
\max\{s + 1, \varepsilon^{-1}\}^{2(s+1)} = \max\{\lambda p + 1, \varepsilon^{-1}\}^{2(\lambda p + 1)} = (\lambda p + 1)^{2(\lambda p + 1)}
\]

which, along with Lemma 8, gives

\[
\| (\mathbf{u} - \mathcal{I}_p \mathbf{u})' \|^2_{0, \Omega} \leq \frac{(p - \lambda p)!}{(p + \lambda p)!} C K^{2(\lambda p + 1)} (\lambda p + 1)^{2(\lambda p + 1)}
\]

\[
\leq \left[ \frac{(1 - \lambda)^{(1-\lambda)}}{(1 + \lambda)^{(1+\lambda)}} \right]^p p^{-2\lambda p} e^{2\lambda p + 1} C K^{2(\lambda p + 1)} (\lambda p + 1)^{2(\lambda p + 1)}
\]

\[
\leq C e K^2 \left[ \frac{(1 - \lambda)^{(1-\lambda)}}{(1 + \lambda)^{(1+\lambda)}} (e K)^{2\lambda} \right]^p (\lambda p + 1)^2 \left( \frac{1 + \lambda p}{p} \right)^{2\lambda p}
\]

\[
\leq C e K^2 \lambda^2 \left[ \frac{(1 - \lambda)^{(1-\lambda)}}{(1 + \lambda)^{(1+\lambda)}} (e K)^{2\lambda} \right]^p \left( \frac{1}{p} + \frac{1}{\lambda p} \right)^{2\lambda p}.
\]

Since \( \left( \frac{1}{p} + \frac{1}{\lambda p} \right)^{2\lambda p} = \lambda^{2\lambda p} \left( \frac{1}{1 + \lambda p} \right)^{2\lambda p} \geq e^{2\lambda^2 p} \), we further get

\[
\| (\mathbf{u} - \mathcal{I}_p \mathbf{u})' \|^2_{0, \Omega} \leq C p^2 \left[ \frac{(1 - \lambda)^{(1-\lambda)}}{(1 + \lambda)^{(1+\lambda)}} (e K)^{2\lambda} \right]^p,
\]

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so if we choose \( \lambda = (eK)^{-1} \in (0, 1) \) we have
\[
\left\| (\vec{u} - I_p \vec{u})' \right\|_{0, \Omega}^2 \leq Cp^2e^{-bp},
\]
(48)
where \( b = |\ln q| \), \( q = \frac{1-\lambda}{1+\lambda} \) \( \in (1, 1) \), and the constant \( C > 0 \) is independent of \( \varepsilon \). Repeating the previous argument for the \( L^2 \) norm of \( (\vec{u} - I_p \vec{u})' \), we get, using (46),
\[
\left\| \vec{u} - I_p \vec{u} \right\|_{0, \Omega} \leq Ce^{-bp},
\]
(49)
with \( C > 0 \) independent of \( \varepsilon \). Combining (48)–(49), and using the definition of the energy norm (14), we get the desired result.

Case 2. \( \kappa \varepsilon < \frac{1}{2} \), i.e. \( p < \frac{1}{2 \kappa \varepsilon} \) (pre-asymptotic case), \( \Delta = \{0, \kappa \varepsilon, 1 - \kappa \varepsilon, 1\} \)

The mesh consists of three elements \( I_i \), \( i = 1, 2, 3 \) and we decompose \( \vec{u} \) as in (20):
\[
\vec{u} = \vec{w} + A^- \vec{u}^- + A^+ \vec{u}^+ + \vec{r}.
\]
The expansion order \( M \) is chosen as the integer part of \( \eta \kappa p \) (and for notational convenience we will simply write \( M = \eta \kappa p \)) where \( \eta > 0 \) is a fixed parameter satisfying
\[
\frac{1}{2} \eta K_2 < 1, \quad \frac{1}{2} \eta K_2 =: \delta < \frac{1}{2}
\]
with \( K_2 \) and \( K_2 \) the constants from Theorems 3 and 5, respectively. The choice of \( \eta \) guarantees that as \( \kappa \varepsilon < \frac{1}{2} \), we have
\[
M \varepsilon K_2 = \eta \kappa p \varepsilon K_2 < \frac{1}{2} \eta K_2 < 1
\]
and
\[
M \varepsilon K_2 = \eta \kappa p \varepsilon K_2 < \frac{1}{2} \eta K_2 = \delta < \frac{1}{2}.
\]
Thus the assumptions of Theorem 3 are satisfied and the remainder \( \vec{r} \) is small by Theorem 5 – in particular, we have
\[
\left\| (\vec{r})^{(n)} \right\|_{0, \Omega} \leq C \varepsilon^{2-n} (\varepsilon K_2 M)^M \leq C \varepsilon^{2-n} \delta^{\eta kp} \leq C \varepsilon^{2-n} e^{-\beta_2 p}, \quad n = 0, 1
\]
(50)
where \( \beta_2 = |\ln q_2| \), \( q_2 = \delta^{\eta k} < 1 \).

We next analyze the approximation of each of the remaining three terms in the decomposition (20).

For the approximation of \( \vec{w} \), we have by Theorem 6 that there exists \( I_p \vec{w} \in \vec{S}(\kappa, p) \) such that \( \vec{w} = I_p \vec{w} \) on \( \partial \Omega \) and for any \( s = 0, 1, \ldots, p \)
\[
\left\| (\vec{w} - I_p \vec{w})' \right\|_{0, \Omega}^2 \leq \frac{(p - s)!}{(p + s)!} \left\| \vec{w}^{(s+1)} \right\|_{0, \Omega}^2 \leq \frac{(p - s)!}{(p + s)!} C K^{2(s+1)} ((s + 1)!)^2,
\]
(51)

\[\]

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Thus, we choose $\lambda = 1/K_2 \in (0, 1)$ and we have
\[
\| (\overline{w} - I_p \overline{w})' \|_{0, \Omega}^2 \leq C p^3 e^{-\beta_2 p},
\]
where $\beta_2 = \ln q_2$, $q_2 = \frac{(1-\lambda)(1-\overline{\lambda})}{(1+\lambda)(1+\overline{\lambda})} < 1$. Repeating the previous argument for the $L^2$ norm of $(\overline{w} - I_p \overline{w})$, we get, using (46),
\[
\| \overline{w} - I_p \overline{w} \|_{0, \Omega}^2 \leq C p e^{-\beta_2 p}.
\]

We now approximate the boundary layers. We will only consider $A^- \overline{w}^-$, since $A^+ \overline{w}^+$ is completely analogous, and in view of (34) and Remark 2, we only need to approximate $u_1^-$. We will construct separate approximations for $u_1^-$ on the intervals $\tilde{I}_1 := I_1 = [0, \kappa \varepsilon]$, and $\tilde{I}_2 := [\kappa \varepsilon, 1]$. By Theorem 6 there exists $I_p u_1^- \in S(\kappa, p)$, such that $I_p u_1^- = u_1^-$ on $\partial \tilde{I}_1$ and for any $s = 0, 1, \ldots, p$
\[
\| (u_1^- - I_p u_1^-)' \|_{0, \tilde{I}_1}^2 \leq (\kappa \varepsilon)^{2s} \frac{(p-s)!}{(p+s)!} \| (u_1^-)_{(s+1)}' \|_{0, \tilde{I}_1}^2.
\]
Now, by Theorem 4, we have
\[
\| (u_1^-)_{(s+1)}' \|_{0, \tilde{I}_1}^2 = \int_0^{\kappa \varepsilon} \left| (u_1^-)_{(s+1)} (x) \right|^2 dx \leq C \kappa \varepsilon K^{2(s+1)} \max \{s + 1, \varepsilon^{-1} \}^{2(s+1)} \max_{x \in [0, \kappa \varepsilon]} \{ e^{-x\alpha/\varepsilon} \}.
\]
Since $\kappa \varepsilon < 1/2$, i.e. $s \leq p < \frac{1}{2 \kappa \varepsilon}$, we have that $\max \{s + 1, \varepsilon^{-1} \}^{2(s+1)} = e^{-2(s+1)}$ and (53), (54) give
\[
\| (u_1^- - I_p u_1^-)' \|_{0, \tilde{I}_1}^2 \leq (\kappa \varepsilon)^{2s} \frac{(p-s)!}{(p+s)!} C \kappa \varepsilon K^{2(s+1)} e^{-2(s+1)} \leq C K^{2(s+1)} \kappa^{2s+1} p^{2s+1} \varepsilon^{s-1} \frac{(p-s)!}{(p+s)!}.
\]
Choosing \( s = \tilde{\lambda}p \) for some \( \tilde{\lambda} \in (0, 1) \), and using Lemma 8, we further obtain

\[
\left\| (u_1^+ - \mathcal{I}_p u_1^-) \right\|_{0, \tilde{I}_2}^2 \leq CK^2(\tilde{\lambda}p + 1)^2 2 \tilde{\lambda}p + 1 2 \tilde{\lambda}p + 1 \varepsilon^{-1} \left| (p - \tilde{\lambda}p)! \right| \left| (p + \tilde{\lambda}p)! \right| 
\leq CK^2(\tilde{\lambda}p + 1)^2 2 \tilde{\lambda}p + 1 2 \tilde{\lambda}p + 1 \varepsilon^{-1} \left\{ 1 - \tilde{\lambda} \right\}^{(1-\lambda)} \left( 1 + \tilde{\lambda} \right)^{(1+\lambda)} \lambda p e^{2 \tilde{\lambda}p + 1} 
\leq CeK^2 \kappa p e^{-1} \left( 1 - \lambda \right)^{1-\lambda} \left( 1 + \lambda \right)^{1+\lambda} (K e K)^2 \tilde{\lambda}p 
\leq C p e^{-1} e^{-\beta_3 p},
\]

where \( \beta_3 = \ln q_3 \), \( q_3 = \frac{(1-\tilde{\lambda})^{(1-\lambda)}}{(1+\lambda)^{(1+\lambda)}} < 1 \), provided we choose \( \kappa = \frac{1}{e K} \). Now, on the interval \( \tilde{I}_2 = [\kappa p e, 1] \), \( u_1^- \) is already exponentially small, and by Theorem 4

\[
\left\| (u_1^-)' \right\|_{0, \tilde{I}_2}^2 = \int_{\kappa p e}^1 \left\| (u_1^-)' \right\| dx \leq C \varepsilon^{-2} (1 - \kappa p e) \max_{x \in \tilde{I}_2} \left\{ e^{-2x\alpha/\varepsilon} \right\} \leq C \varepsilon^{-2} e^{-2\kappa p \alpha}.
\]

Thus, we approximate \( u_1^- \) by its linear interpolant \( \mathcal{I}_1 u_1^- \), and we have

\[
\left\| (u_1^- - \mathcal{I}_1 u_1^-)' \right\|_{0, \tilde{I}_2}^2 \leq \left\| (u_1^-)' \right\|_{0, \tilde{I}_2}^2 + \left\| (\mathcal{I}_1 u_1^-)' \right\|_{0, \tilde{I}_2}^2 \leq C \varepsilon^{-2} e^{-2\kappa p \alpha},
\]

which along with (55) give

\[
\left\| (u_1^- - \mathcal{I}_p u_1^-) \right\|_{0, \Omega}^2 \leq C p e^{-2} e^{-\beta_4 p},
\]

for some \( \beta_4 > 0 \) independent of \( \varepsilon \). Repeating the previous arguments for the \( L^2 \) norm of \( (u_1^- - \mathcal{I}_p u_1^-) \), we get

\[
\left\| u_1^- - \mathcal{I}_p u_1^- \right\|_{0, \Omega}^2 \leq C e^{-\beta_5 p},
\]

for some \( \beta_5 > 0 \), independent of \( \varepsilon \). Using the same techniques, similar bounds can be obtained for \( \overline{u}_+ \).

Combining (50), (52), (56), (57) and the analogous bounds for \( \overline{u}_+ \), we have

\[
\left\| \overline{u} - \mathcal{I}_p \overline{u} \right\|_{0, \Omega}^2 = \left\| \left( \overline{w} + \left[ -w_1(0)\overline{u}_1^- \right] + \left[ -w_1(1)\overline{u}_1^+ \right] + \overline{\tau} \right) - ( \mathcal{I}_p \overline{w} + \left[ -w_1(0)\mathcal{I}_p u_1^- \right] + \left[ -w_1(1)\mathcal{I}_p u_1^+ \right] ) \right\|_{0, \Omega}^2 
\leq \left\| \overline{w} - \mathcal{I}_p \overline{w} \right\|_{0, \Omega}^2 + C \left\{ \left\| u_1^- - \mathcal{I}_p u_1^- \right\|_{0, \Omega}^2 + \left\| u_1^+ - \mathcal{I}_p u_1^+ \right\|_{0, \Omega}^2 \right\} + \left\| \overline{\tau} \right\|_{0, \Omega}^2 
\leq C p e^{-\beta_3 p},
\]

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for some $\beta > 0$, independent of $\varepsilon$. Similarly,

$$
|u_1 - I_p u_1|_{1,\Omega}^2 \leq |w_1 - I_p w_1|_{1,\Omega}^2 + C \left\{ |u_1^- - I_p u_1^-|_{1,\Omega}^2 + |u_1^+ - I_p u_1^+|_{1,\Omega}^2 \right\} + |r_1|_{1,\Omega}^2
$$

and

$$
|u_2 - I_p u_2|_{1,\Omega}^2 \leq |w_1 - I_p w_1|_{1,\Omega}^2 + |r_2|_{1,\Omega}^2 \leq C \varepsilon^{-2} p^3 e^{-\beta p},
$$

so that

$$
\|\overline{u} - I_p \overline{u}\|_{E,\Omega}^2 = \varepsilon^2 |u_1 - I_p u_1|_{1,\Omega}^2 + |u_2 - I_p u_2|_{1,\Omega}^2 + \alpha^2 \|\overline{u} - I_p \overline{u}\|_{0,\Omega}^2 \leq C p^3 e^{-\beta p}
$$
as desired.

Using the above theorem and the quasioptimality result (18) we have the following.

**Corollary 10.** Let $\overline{u}$ be the solution to (4)–(5) and let $\overline{u}_{FE} \in S_p^0(\Delta)$ be the solution to (17). Then there exist constants $\kappa, C, \sigma > 0$ depending only on the input data $\overline{f}$ and $A$ such that

$$
\|\overline{u} - \overline{u}_{FE}\|_{E,\Omega} \leq C \varepsilon^{3/2} e^{-\sigma p}.
$$

The above result shows that as $p \to \infty$ the method converges at an exponential rate, independently of $\varepsilon$, when the error is measured in the energy norm. In the numerical study from [20] it was observed that the method not only converges at the above rate, but as $\varepsilon \to 0$ the performance improves. In particular, the following estimate was observed:

$$
\|\overline{u} - \overline{u}_{FE}\|_{E,\Omega} \leq C \varepsilon^{1/2} e^{-\sigma p}.
$$

This was the case for the scalar problem with constant coefficients and polynomial right hand side studied in [16]. It is interesting to note that in [20] the above rate was observed for the variable coefficient problem as well, even though for this case no exact solution was available and the errors were computed using a reference solution. These remarks will be illustrated in the next section.

## 4 Numerical Results

In this section we present the results of numerical computations for the two model problems considered in [3] and [6], having as our goal the illustration our theoretical findings; we refer the interested reader to [20] for a detailed numerical study on these problems. We should mention that the constant $\kappa$ in the three element mesh is taken to be 1 for all the computations performed.

### 4.1 The constant coefficient case

First we consider the constant coefficient case, in which

$$
A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad \overline{f}(x) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \overline{u}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
$$
An exact solution is available, hence the computations we report are reliable. We will be plotting the percentage relative error in the energy norm, given by

\[ 100 \times \frac{\| \overline{u}_{EXACT} - \overline{u}_{FEM} \|_{E,\Omega}}{\| \overline{u}_{EXACT} \|_{E,\Omega}}, \]  

versus the number of degrees of freedom \( N \), on log-log and semi-log scales. Figure 2 shows the performance of the \( hp \) version on the 3 element mesh for different values of \( \varepsilon \), and, as mentioned before, we observe that the method not only does not deteriorate as \( \varepsilon \to 0 \), but it actually performs better, when the error is measured in the energy norm. This is reflected by the positive powers of \( \varepsilon \) in the error estimate (58). We also note that when \( \varepsilon = 10^{-j}, j = 5,6 \) the error initially decreases in an awkward fashion, but as \( p \to \infty \) it goes down at the (expected) exponential rate. We believe this is due to the fact that the same (3 element) mesh is used to approximate both components of the solution, even though the second component does not have a boundary layer and the use of the 3 element mesh might not be the ‘best’ choice, especially when the polynomial degree \( p \) is small. We do not wish to dwell on this, since even with the use of the same mesh for both components, the error eventually decreases exponentially fast as \( p \to \infty \).

4.2 The variable coefficient case

Next, we consider the variable coefficient case, in which

\[ A = \begin{bmatrix} 2(x + 1)^2 & -(1 + x^2) \\ -2 \cos(\pi x/4) & 2e^{1-x} \end{bmatrix}, \quad f(x) = \begin{bmatrix} 2e^x \\ 10x + 1 \end{bmatrix}, \quad \overline{u}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \]

An exact solution is not available, and for our computations we use a reference solution obtained with a large number of degrees of freedom on a very fine mesh which includes exponential refinement near the endpoints of the domain (see [20] for more details). We are again interested in the (now estimated) percentage relative error in the energy norm.

Figure 3 shows the performance of the \( hp \) version on the 3 element mesh for different values of \( \varepsilon \), and we observe that the results are almost identical to those obtained for the constant coefficient problem, namely the method converges at an exponential rate as \( p \to \infty \), independently of \( \varepsilon \), with improved performance as \( \varepsilon \to 0 \).

5 Concluding Remarks

We have studied the approximation of a weakly coupled system of singularly perturbed reaction-diffusion equations, by the finite element method. We showed that under the assumption of analytic input data, the \( hp \) version on the variable three element mesh \( \{0, \kappa p \varepsilon, 1 - \kappa p \varepsilon, 1\} \) yields exponential convergence, independently of \( \varepsilon \), as \( p \to \infty \), when the error is measured in the energy norm. The constant \( \kappa \) in the mesh was shown to depend on the constant of analyticity of the input data. The present article, in conjunction with [21], [22], complete the \( hp \) FEM analysis (and proof of robust exponential convergence) for the system (1), (3).
Figure 2: Energy norm convergence for the $hp$ version. Top: Log-log scale. Bottom: Semi-log scale
Figure 3: Energy norm convergence for the $hp$ version. Top: Log-log scale. Bottom: Semi-log scale
References


