PAMM · Proc. Appl. Math. Mech. 7, 2020055–2020056 (2007) / DOI 10.1002/pamm.200700348

An hp finite element method for singularly perturbed systems of reaction-diffusion equations

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We consider the approximation of a coupled system of two singularly perturbed reaction-diffusion equations by the finite element method. The solution to such problems contains boundary layers which overlap and interact, and the numerical approximation must take this into account in order for the resulting scheme to converge uniformly with respect to the singular perturbation parameters. We present results on a high order hp finite element scheme which includes elements of size $O(\varepsilon p)$ and $O(\mu p)$ near the boundary, where ε, μ are the singular perturbation parameters and p is the degree of the approximating polynomials. Under the assumption of analytic input data, the method yields *exponential* rates of convergence as $p \to \infty$, independently of ε and μ.

1 The Model Problem and its Formulation

We consider the following system of two singularly perturbed reaction diffusion equations: Find $\overrightarrow{u}(x) = [u_1(x), u_2(x)]^T$ such that

$$L\overrightarrow{u} := \begin{bmatrix} -\varepsilon^2 \frac{d^2}{dx^2} & 0 \\ 0 & -\mu^2 \frac{d^2}{dx^2} \end{bmatrix} \overrightarrow{u} + A\overrightarrow{u} = \overrightarrow{f} \text{ in } \Omega = (0,1), \quad \overrightarrow{u}(0) = \overrightarrow{u}(1) = \overrightarrow{0}, \tag{1}$$

where $0 < \varepsilon \leq \mu < 1,\ A = \begin{bmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{bmatrix}$ and $\overrightarrow{f}(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix}$ are given. The assumptions on the data are that

$$a_{12}(x) \leq 0, \quad a_{21}(x) \leq 0 \quad \forall\, x \in \overline{\Omega}, \quad \min_{\overline{\Omega}} \{a_{11}(x) + a_{12}(x),\ a_{21}(x) + a_{22}(x)\} \geq \alpha^2 > 0, \tag{2}$$

for some $\alpha \in \mathbb{R}$, and that there exist constants $C_f, \gamma_f, C_a, \gamma_a > 0$ such that for $i, j = 1, 2$ and $\forall\, n \in \mathbb{N}_0$

$$\left\| f_i^{(n)} \right\|_{L^\infty(\Omega)} \leq C_f \gamma_f^n n! \quad , \quad \left\| a_{ij}^{(n)} \right\|_{L^\infty(\Omega)} \leq C_a \gamma_a^n n!. \tag{3}$$

The presence of ε and μ in (1) causes the solution $\overrightarrow{u}$ to have boundary layers near the endpoints of Ω, which, in general, overlap and interact. Problems of this type arise in the modelling of turbulance in water waves [2], as well as in the finite element approximation of shells, where the singular perturbation parameters are related to the thickness t of the shell; for example, in Naghdi-type thin shell models in mechanics there is an $O(t)$ layer due to shear deformation and there is a second layer (or length scale) $O(t^\beta)$, with $\beta \in \{1/2, 1/3, 1/4\}$ (depending on the principal curvatures of the shell's midsurface), due to bending and membrane coupling [1]. The 2-scale reaction-diffusion system (1) could be considered as a model problem for this situation, with $\varepsilon = t$ and $\mu = t^\beta$.

The variational formulation of (1) reads: Find $\overrightarrow{u} \in \left[H_0^1(\Omega)\right]^2$ such that $B(\overrightarrow{u}, \overrightarrow{v}) = F(\overrightarrow{v}) \ \ \forall\, \overrightarrow{v} \in \left[H_0^1(\Omega)\right]^2$, where, with $(u,v)_\Omega := \int_\Omega uv$,

$$B(\overrightarrow{u}, \overrightarrow{v}) = \varepsilon^2 (u_1', v_1')_\Omega + \mu^2 (u_2', v_2')_\Omega + \sum_{i=1}^{2}\sum_{j=1}^{2} (a_{ij} u_j, v_i)_\Omega \ , \ F(\overrightarrow{v}) = \sum_{i=1}^{2} (f_i, v_i)_\Omega \,. \tag{4}$$

The unique solvability of the variational problem follows from the continuity of $B(\cdot,\cdot)$ and $F(\cdot)$, and from the coercivity of $B(\cdot,\cdot)$ with respect to the *energy norm* $\|\overrightarrow{u}\|_{E,\Omega}^2 := \varepsilon^2 |u_1|_{H^1(\Omega)}^2 + \mu^2 |u_2|_{H^1(\Omega)}^2 + \alpha^2 \left(\|u_1\|_{L^2(\Omega)}^2 + \|u_2\|_{L^2(\Omega)}^2 \right)$. For the discretization, we choose a finite dimensional subspace S_N of $H_0^1(\Omega)$ and solve the problem: Find $\overrightarrow{u}_N \in [S_N]^2$ such that

$$B(\overrightarrow{u}_N, \overrightarrow{v}) = F(\overrightarrow{v}) \ \ \forall\, \overrightarrow{v} \in [S_N]^2 \,. \tag{5}$$

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It can be shown that the discrete problem (5) admits a unique solution, which, by the well-known orthogonality relation, is the best approximation from the space $[S_N]^2$.

Now, by the analyticity of a_{ij} and f_i, we have that u_i are analytic and that there exist constants C and $K > 0$, independent of ε and μ, such that $\left| u_i^{(n)} \right|_{0,\Omega} \leq CK^n \max\{n, \varepsilon^{-1}\}^n \quad \forall\, n \in \mathbb{N}_0,\ i = 1, 2$. This result is not sufficient for the analysis of our method, hence we can show that the solution $\overrightarrow{u}$ of (1) can be decomposed as

$$\overrightarrow{u} = \overrightarrow{w} + A^- \overrightarrow{u}^- + A^+ \overrightarrow{u}^+ + \overrightarrow{r}, \tag{6}$$

where $\overrightarrow{w}$ is the smooth part, $\overrightarrow{u}^{\pm}$ are the two boundary layer parts (with $A^{\pm}$ appropriately chosen constant matrices) and $\overrightarrow{r}$ is the smooth remainder. The nature of each term in the above decomposition depends on the relationship between ε and μ, and is discussed in detail in [3]–[5], where specific bounds are obtained on the derivatives of each term which show that indeed $\overrightarrow{w}$ and $\overrightarrow{r}$ are smooth – the bounds on the derivatives of the boundary layer parts depend, as expected, on the singular perturbation parameters.

2 The Finite Element Method

Let $\Delta = \{0 = x_0 < x_1 < ... < x_{\mathcal{M}} = 1\}$ be an arbitrary partition of $\Omega = (0, 1)$ and set $I_j = (x_{j-1}, x_j)$, $h_j = x_j - x_{j-1}$, $j = 1, ..., \mathcal{M}$. Also, define the master (or standard) element $I_{ST} = (-1, 1)$, and note that it can be mapped onto the j^{th} element I_j by the linear mapping $x = Q_j(t) = \frac{1}{2}(1-t)\, x_{j-1} + \frac{1}{2}(1+t)\, x_j$. With $\Pi_p(I_{ST})$ the space of polynomials of degree $\leq p$ on I_{ST}, we define our finite dimensional subspaces as

$$S_N \equiv S^{\overrightarrow{p}}(\Delta) = \left\{ u \in H_0^1(\Omega) : u(Q_j(t)) \in \Pi_{p_j}(I_{ST}), j = 1, ..., \mathcal{M} \right\} \quad , \quad \overrightarrow{S}_0^p(\Delta) := [S^{\overrightarrow{p}}(\Delta) \cap H_0^1(\Omega)]^2, \tag{7}$$

where $\overrightarrow{p} = (p_1, ..., p_{\mathcal{M}})$ is the vector of polynomial degrees assigned to the elements.

We now describe the mesh used for the method: If we are in the asymptotic range of p, i.e. $p \geq 1/\varepsilon \geq 1/\mu$, then a single element suffices since p will be sufficiently large to give us exponential convergence without any refinement. If we are in the pre-asymptotic range of p then the mesh consists of either five or three elements as described below.

Definition 2.1 For $\kappa > 0$, $p \in \mathbb{N}$ and $0 < \varepsilon \leq \mu \leq 1$, define the spaces $\overrightarrow{S}(\kappa, p)$ of piecewise polynomials by

$$\overrightarrow{S}(\kappa, p) := \begin{cases} \overrightarrow{S}_0^p(\Delta);\ \Delta = \{0, 1\} & \text{if } \kappa p \varepsilon \geq \frac{1}{2}, \\ \overrightarrow{S}_0^p(\Delta);\ \Delta = \{0, \kappa p\varepsilon, \kappa p\mu, 1-\kappa p\mu, 1-\kappa p\varepsilon, 1\} & \text{if } \kappa p\mu < \frac{1}{2}, \\ \overrightarrow{S}_0^p(\Delta);\ \Delta = \{0, \kappa p\varepsilon, 1-\kappa p\varepsilon, 1\} & \text{if } \kappa p\varepsilon < \frac{1}{2} \ \&\ \kappa p\mu \geq \frac{1}{2}, \text{ or if } \varepsilon = \mu \ \&\ \kappa p\varepsilon < \frac{1}{2}, \\ & \text{or if } \mu = 1 \ \&\ \kappa p\varepsilon < \frac{1}{2} \end{cases}$$

In all cases above the polynomial degree is uniformly p on all elements.

We now state our main result.

Theorem 2.2 *Let $\overrightarrow{f}$ and A be composed of functions that are analytic on $\overline{\Omega}$ and satisfy the conditions in* (2)–(3). *Let $\overrightarrow{u} = [u_1, u_2]^T$ be the solution to* (1). *Then there exist constants $\kappa, C, \beta > 0$ depending only on $\overrightarrow{f}$ and A such that there exists $\mathcal{I}_p \overrightarrow{u} = [\mathcal{I}_p u_1, \mathcal{I}_p u_2]^T \in \overrightarrow{S}(\kappa, p)$ with $\mathcal{I}_p \overrightarrow{u} = \overrightarrow{u}$ on $\partial\Omega$ and*

$$\|\overrightarrow{u} - \mathcal{I}_p \overrightarrow{u}\|_{E,\Omega}^2 \leq Cp^3 e^{-\beta p}.$$

From the above theorem and the best approximation result, we have the following.

Corollary 2.3 *Let $\overrightarrow{u}$ be the solution to* (1) *and let $\overrightarrow{u}_{FE} \in \overrightarrow{S}_0^p(\Delta)$ be the solution to* (5). *Then exist constants $\kappa, C, \sigma > 0$ depending only on the input data $\overrightarrow{f}$ and A such that*

$$\|\overrightarrow{u} - \overrightarrow{u}_{FE}\|_{E,\Omega} \leq Cp^{3/2} e^{-\sigma p}.$$

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