An Overview of Robot-Sensor Calibration Methods for Evaluation of Perception Systems

Mili Shah
Loyola University Maryland
4501 North Charles Street
Baltimore, MD, 21210
1-410-617-2724
mishah@loyola.edu

Roger D. Eastman
Loyola University Maryland
4501 North Charles Street
Baltimore, MD, 21210
1-410-617-2281
reastman@loyola.edu

Tsai Hong
National Institute of Standards and Technology
100 Bureau Drive
Gaithersburg, MD, 20899
1-301-975-3444
hongt@nist.gov

ABSTRACT

In this paper, an overview of methods that solve the robot-sensor calibration problem of the forms $\mathbf{AX} = \mathbf{XB}$ and $\mathbf{AX} = \mathbf{YB}$ is given. Each form will be split into three solutions: separable closed-form solutions, simultaneous closed-form solutions, and iterative solutions. The advantages and disadvantages of each of the solutions in the case of evaluation of perception systems will also be discussed.

Categories and Subject Descriptors

C.4 [**Performance of Systems**]: Performance attributes; B.8.2 [**Performance and Reliability**]: Performance Analysis and Design Aids; G.1.6 [**Optimization**]: Global optimization; I.4.8 [**Scene Analysis**]: Motion, Tracking; I.5.4 [**Applications**]: Computer Vision

General Terms

Computer Vision, Robot-Sensor Calibration, Hand-Eye Calibration, Performance Evaluation

1. INTRODUCTION

Robot-sensor calibration has been an active area of research for many decades. The most common mathematical representations for the robot-sensor calibration problem consist of two forms: $\mathbf{AX} = \mathbf{XB}$ and $\mathbf{AX} = \mathbf{YB}$. Examples for each of the forms can be seen in Figure 1. Specifically in Figure 1a, $\mathbf{A}_i$ represents robot motion, $\mathbf{B}_i$ represents camera motion, and the unknown $\mathbf{X}$ represents the fixed homogeneous transformation between the robot base and camera. Following the arrows, it can easily be seen that

$$\mathbf{A}_i\mathbf{X} = \mathbf{X}\mathbf{B}_i \Rightarrow \mathbf{AX} = \mathbf{XB},$$

where $\mathbf{A} = \mathbf{A}_i$ and $\mathbf{B} = \mathbf{B}_i$. Similarly in Figure 1b, $\mathbf{A}_i$ represents the transformation from robot base to gripper, $\mathbf{B}_i$ represents the transformation from camera to object, and the unknown $\mathbf{X}$ represents the fixed homogeneous transformation between gripper and camera. Following the arrows

$$\mathbf{A}_1\mathbf{X}\mathbf{B}_1 = \mathbf{A}_2\mathbf{X}\mathbf{B}_2 \Leftrightarrow \mathbf{A}_2^{-1}\mathbf{A}_1\mathbf{X} = \mathbf{X}\mathbf{B}_2\mathbf{B}_1^{-1} \Rightarrow \mathbf{AX} = \mathbf{XB},$$

where $\mathbf{A} = \mathbf{A}_2^{-1}\mathbf{A}_1$ and $\mathbf{B} = \mathbf{B}_2\mathbf{B}_1^{-1}$. Finally in Figure 1c, $\mathbf{A}_i$ represents the transformation from target to sensor, $\mathbf{B}_i$ represents the transformation from camera to object, the unknown $\mathbf{X}$ represents the fixed homogeneous transformation between sensor and object, and the unknown $\mathbf{Y}$ represents the fixed homogeneous transformation between target and camera. Following the arrows

$$\mathbf{A}_i\mathbf{X} = \mathbf{Y}\mathbf{B}_i \Rightarrow \mathbf{AX} = \mathbf{YB},$$

where $\mathbf{A} = \mathbf{A}_i$ and $\mathbf{B} = \mathbf{B}_i$.

In this paper, we will give an overview of methods to solve $\mathbf{AX} = \mathbf{XB}$ and $\mathbf{AX} = \mathbf{YB}$. Notice that for

$$\begin{aligned} \mathbf{AX} &= \mathbf{XB} \\ \begin{pmatrix} \mathbf{R_A} & \mathbf{t_A} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \mathbf{R_X} & \mathbf{t_X} \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} \mathbf{R_X} & \mathbf{t_X} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \mathbf{R_B} & \mathbf{t_B} \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} \mathbf{R_A}\mathbf{R_X} & \mathbf{R_A}\mathbf{t_X} + \mathbf{t_A} \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} \mathbf{R_X}\mathbf{R_B} & \mathbf{R_X}\mathbf{t_B} + \mathbf{t_X} \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

Thus,

$$\mathbf{R_A}\mathbf{R_X} = \mathbf{R_X}\mathbf{R_B},$$

which we will define as the *orientational component*, and

$$\mathbf{R_A}\mathbf{t_X} + \mathbf{t_A} = \mathbf{R_X}\mathbf{t_B} + \mathbf{t_X}.$$

which we will define as the *positional component* for $\mathbf{AX} = \mathbf{XB}$. The orientational component

$$\mathbf{R_A}\mathbf{R_X} = \mathbf{R_Y}\mathbf{R_B},$$

and positional component

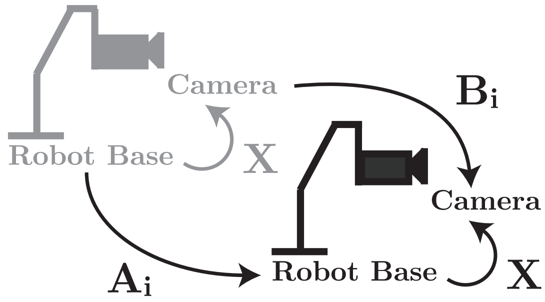
$$\mathbf{R_A}\mathbf{t_X} + \mathbf{t_A} = \mathbf{R_Y}\mathbf{t_B} + \mathbf{t_Y}$$

for $\mathbf{AX} = \mathbf{YB}$ can similarly be constructed. The methods to solve $\mathbf{AX} = \mathbf{XB}$ and $\mathbf{AX} = \mathbf{YB}$ consist of three forms: separable closed-form solutions, simultaneous closed-form solutions, and iterative closed-form solutions. The separable closed-form solutions arise from solving the orientational component separately from the positional component, the simultaneous closed-form solutions arise from simultaneously solving the orientational component and the positional component, while the iterative solutions arise from solving both the orientational component and positional component iteratively using optimization techniques. Details of each of the

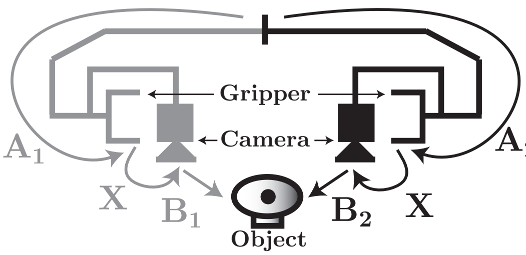
(c) 2012 Association for Computing Machinery. ACM acknowledges that this contribution was authored or co-authored by a contractor or affiliate of the U.S. Government. As such, the Government retains a nonexclusive, royalty-free right to publish or reproduce this article, or to allow others to do so, for Government purposes only.

PerMIS '12, March 20-22, 2012, College Park, MD, USA.

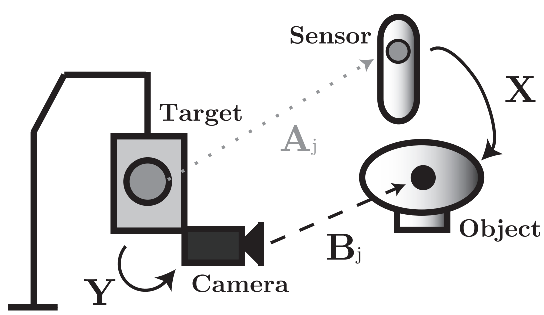
Copyright © 2012 ACM 978-1-4503-1126-7-3/22/12 ...$10.00.



(a) $\mathbf{AX} = \mathbf{XB}$ where the unknown $\mathbf{X}$ represents the transformation from robot base to camera.



(b) $\mathbf{AX} = \mathbf{XB}$ where the unknown $\mathbf{X}$ represents the transformation from gripper to camera.



(c) $\mathbf{AX} = \mathbf{YB}$ where the unknown $\mathbf{X}$ represents the transformation from sensor to object and the unknown $\mathbf{Y}$ represents the transformation from target to camera.

Figure 1: Different experimental setups for robot-sensor calibration.

solutions will be discussed in the following sections. Specifically for $\mathbf{AX} = \mathbf{XB}$, separable closed-form solutions will be discussed in Section 2.1, simultaneous closed-form solutions will be discussed in Section 2.2, and iterative solutions will be discussed in Section 2.3. Following, in Section 3, will be a section discussing the different solutions for $\mathbf{AX} = \mathbf{YB}$. Finally, concluding remarks, which will include the advantages and disadvantages for each of the solutions in the evaluation of perception systems, will be discussed in Section 4.

2. AX=XB SOLUTIONS

2.1 Separable Solutions for AX=XB

The robot-sensor calibration problem of the form $\mathbf{AX} = \mathbf{XB}$ was introduced in the work of Shiu and Ahmad [21]. In this paper, they solve the robot-sensor calibration problem by separating the problem into its orientational component

$$\mathbf{R_A R_X} = \mathbf{R_X R_B}$$

and positional component

$$\mathbf{R_A t_X} + \mathbf{t_A} = \mathbf{R_X t_B} + \mathbf{t_X}.$$

They solve the orientational component by utilizing the angle-axis formulation of rotation; i.e., let $\mathbf{R} = \mathrm{Rot}(k_{\mathbf{R}}, \theta)$, where $k_{\mathbf{R}}$ is the axis of rotation of $\mathbf{R}$ and θ is the angle. Specifically, they state that the general solution

$$\mathbf{R_X} = \mathrm{Rot}(k_{\mathbf{A}_i}, \beta_i)\mathbf{R}_{\mathbf{X}_{P_i}},$$

where

$$\begin{aligned} \mathbf{R}_{\mathbf{X}_{P_i}} &= \mathrm{Rot}(\mathbf{v}, \omega) \\ \mathbf{v} &= k_{\mathbf{B}_i} \times k_{\mathbf{A}_i} \\ \omega &= \mathrm{atan2}(|k_{\mathbf{B}_i} \times k_{\mathbf{A}_i}|, k_{\mathbf{B}_i} \cdot k_{\mathbf{A}_i}) \end{aligned}$$

and β_i is calculated by solving a $9 \times 2n$ linear system of equations where the number of frames $n \geq 2$. They also prove for uniqueness at least two of the axes of rotation of $\mathbf{R}_{\mathbf{A}_i}$ cannot be parallel. Once $\mathbf{R_X}$ is formulated, the positional component

$$\begin{pmatrix} \mathbf{R}_{A_1} - \mathbf{I} \\ \vdots \\ \mathbf{R}_{A_n} - \mathbf{I} \end{pmatrix} \mathbf{t_X} = \begin{pmatrix} \mathbf{R_X t}_{\mathbf{B}_1} - \mathbf{t}_{\mathbf{A}_1} \\ \vdots \\ \mathbf{R_X t}_{\mathbf{B}_n} - \mathbf{t}_{\mathbf{A}_n} \end{pmatrix}$$

can be solved using standard linear system techniques. This is the general technique of separable solutions for $\mathbf{AX} = \mathbf{XB}$: first calculate $\mathbf{R_X}$ using some technique and then use that $\mathbf{R_X}$ to solve for $\mathbf{t_X}$ using standard linear system techniques. Thus, for the rest of this section concentration will be placed solely on calculating the optimal rotation $\mathbf{R_X}$.

A problem with the Shiu and Ahmad method is that the size of the linear system doubles each time a new frame is added to the system. An alternative method by Tsai and Lenz [23] solves the robot-sensor calibration method using a fixed size linear system. The derivation is simpler than the Shiu and Ahmad method and computationally more efficient. Specifically, Tsai and Lenz solve the orientational component by again considering the angle-axis formulation $\mathbf{R} = \mathrm{Rot}(k_{\mathbf{R}}, \theta)$ for rotation. They find the axis of rotation $k_{\mathbf{R_X}}$ for $\mathbf{R_X}$ by solving

$$\begin{aligned} \mathrm{Sk}\left(k_{\mathbf{R}_{A_i}} + k_{\mathbf{R}_{B_i}}\right) k'_{\mathbf{R_X}} &= k_{\mathbf{R}_{A_i}} - k_{\mathbf{R}_{B_i}} \\ k_{\mathbf{R_X}} &= \frac{2k'_{\mathbf{R_X}}}{\sqrt{1 + \left|k'_{\mathbf{R_X}}\right|^2}} \end{aligned} \quad (1)$$

where the skew-symmetric matrix

$$\mathrm{Sk}(\mathbf{x}) = \begin{pmatrix} 0 & -\mathbf{x}(3) & \mathbf{x}(2) \\ \mathbf{x}(3) & 0 & -\mathbf{x}(1) \\ -\mathbf{x}(2) & \mathbf{x}(1) & 0 \end{pmatrix},$$

and the angle of rotation θ for $\mathbf{R_X}$ by setting

$$\theta = 2\mathrm{atan}\left|k'_{\mathbf{R_X}}\right|.$$

Another formulation that utilizes the angle-axis formulation was presented by Wang in [24]. They solve the orientational component by considering the properties of the axes of rotation of $\mathbf{R}_{\mathbf{A}_i}$, $\mathbf{R}_{\mathbf{B}_i}$, $\mathbf{R}_{\mathbf{A}_{i+1}}$, and $\mathbf{R}_{\mathbf{B}_{i+1}}$ for $i = 1, 2, \ldots n-1$. Wang compares his method with the Shiu and

Ahmad method [21] and the Tsai and Lenz method [23]. He concludes that of the three methods, the Tsai and Lenz method is the best on average.

The angle-axis methods for calculating the solution of the robot-sensor calibration problem up to this point can be cumbersome. In order to simplify the problem, Park and Martin formed a solution for $\mathbf{R_X}$ by taking advantage of Lie group theory to transform the orientational component into a linear system [17]. Specifically, they take advantage of the property that for a given rotation matrix $\mathbf{R}$

$$\log \mathbf{R} = \frac{\theta}{2\sin\theta}\left(\mathbf{R} - \mathbf{R}^T\right) = \mathrm{Sk}(\mathbf{r}).$$

Here, $\mathbf{r} = \theta k_{\mathbf{R}}$ where θ is the angle of rotation of $\mathbf{R}$ and $k_{\mathbf{R}}$ is the axis of rotation of $\mathbf{R}$. For this paper, $\mathbf{r}$ is the shorthand notation of $\log \mathbf{R}$. Using this formulation,

$$\mathbf{R}_{\mathbf{A}_i}\mathbf{R_X} = \mathbf{R_X}\mathbf{R}_{\mathbf{B}_i} \quad \Leftrightarrow \quad \mathbf{R_X}\mathbf{a}_i = \mathbf{b}_i$$

where $\mathbf{a}_i$ and $\mathbf{b}_i$ are the shorthand logarithms of $\mathbf{A}_i$ and $\mathbf{B}_i$, respectively. In the presence of noise, Park and Martin calculate the solution of the robot-sensor problem by solving

$$\min_{\mathbf{R_X}} \sum_{i=1}^{n} \|\mathbf{R_X}\ \mathbf{a}_i - \mathbf{b}_i\|^2,$$

whose closed-form solution can be calculated efficiently as

$$\mathbf{R_X} = \mathbf{U}\mathbf{V}^{-1/2}\mathbf{U}^{-1}\mathbf{M}^T$$

where $\mathbf{M} = \sum_{i=1}^{n} \mathbf{b}_i\mathbf{a}_i^T$ and the eigendecomposition of $\mathbf{M}^T\mathbf{M} = \mathbf{U}\mathbf{V}\mathbf{U}^{-1}$.

Chou and Kamel introduce quaternions into the robot-sensor calibration problem in [4, 5]. They notice that the orientational component

$$\mathbf{R_A}\mathbf{R_X} = \mathbf{R_X}\mathbf{R_B} \quad \Leftrightarrow \quad \mathbf{q_A} * \mathbf{q_X} = \mathbf{q_X} * \mathbf{q_B}$$

where $\mathbf{q_X}$ is the quaternion representation of the rotation matrix $\mathbf{R_X}$. Using the matrix form of quaternion multiplication, the orientational component can be restructured into a linear system

$$\begin{aligned} \mathbf{q_A} * \mathbf{q_X} - \mathbf{q_X} * \mathbf{q_B} &= \mathbf{q_A} * \mathbf{q_X} - \overline{\mathbf{q_B}} * \mathbf{q_X} \\ &= (\mathbf{q_A} - \overline{\mathbf{q_B}}) * \mathbf{q_X} = 0 \end{aligned}$$

since

$$\begin{aligned} \mathbf{q_X} * \mathbf{q_B} &= \begin{pmatrix} x_0 & -\mathbf{x}^T \\ \mathbf{x} & (x_0\mathbf{I} + \mathrm{Sk}(\mathbf{x})) \end{pmatrix} \begin{pmatrix} b_0 \\ \mathbf{b} \end{pmatrix} \\ &= \begin{pmatrix} x_0 b_0 - \mathbf{x}^T\mathbf{b} \\ \mathbf{x}b_0 + (x_0\mathbf{I} + \mathrm{Sk}(\mathbf{x}))\,\mathbf{b} \end{pmatrix} \\ &= \begin{pmatrix} b_0 x_0 - \mathbf{b}^T\mathbf{x} \\ \mathbf{b}x_0 + (b_0\mathbf{I} - \mathrm{Sk}(\mathbf{b}))\,\mathbf{x} \end{pmatrix} \\ &= \begin{pmatrix} b_0 & -\mathbf{b}^T \\ \mathbf{b} & (b_0\mathbf{I} - \mathrm{Sk}(\mathbf{b})) \end{pmatrix} \begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} \\ &= \overline{\mathbf{q_B}} * \mathbf{q_X}. \end{aligned}$$

Chou and Kamel solve the linear system using the singular value decomposition.

Horaud and Dornaika form another closed-form solution for $\mathbf{R_X}$ via quaternions in [11]. Specifically, they find that the quaternion representation $\mathbf{q_X}$ for $\mathbf{R_X}$ can be found as the eigenvector associated with the smallest (positive) eigenvalue of

$$\mathcal{A} = \sum_{i=1}^{n} \mathcal{A}_i^T \mathcal{A}_i$$

where

$$\mathcal{A}_i = \begin{pmatrix} 0 & -\mathbf{a}_x^{(i)} + \mathbf{b}_x^{(i)} & -\mathbf{a}_y^{(i)} + \mathbf{b}_y^{(i)} & -\mathbf{a}_z^{(i)} + \mathbf{b}_z^{(i)} \\ \mathbf{a}_x^{(i)} - \mathbf{b}_x^{(i)} & 0 & -\mathbf{a}_z^{(i)} - \mathbf{b}_z^{(i)} & \mathbf{a}_y^{(i)} + \mathbf{b}_y^{(i)} \\ \mathbf{a}_y^{(i)} - \mathbf{b}_y^{(i)} & \mathbf{a}_z^{(i)} + \mathbf{b}_z^{(i)} & 0 & -\mathbf{a}_x^{(i)} - \mathbf{b}_x^{(i)} \\ \mathbf{a}_z^{(i)} - \mathbf{b}_z^{(i)} & -\mathbf{a}_y^{(i)} - \mathbf{b}_y^{(i)} & \mathbf{a}_x^{(i)} + \mathbf{b}_x^{(i)} & 0 \end{pmatrix}$$

and $\mathbf{a}^{(i)} = (\mathbf{a}_x^{(i)}, \mathbf{a}_y^{(i)}, \mathbf{a}_z^{(i)})^T$ is the axis of rotation for $\mathbf{A}_i$ and $\mathbf{b}^{(i)} = (\mathbf{b}_x^{(i)}, \mathbf{b}_y^{(i)}, \mathbf{b}_z^{(i)})^T$ is the axis of rotation for $\mathbf{B}_i$.

Zhuang and Roth also apply quaternions to the robot-sensor calibration problem in [28] to get a closed-form solution that is very similar in formulation to the angle-axis formulation (1) of Tsai and Lenz [23].

Liang et al. apply the Kronecker product to the orientational component of the robot-sensor problem to solve for $\mathbf{R_X}$ in [14]. As a result, the orientational component becomes the linear system

$$\underbrace{\begin{pmatrix} \mathbf{R}_{\mathbf{A}_1} \otimes \mathbf{I} - \mathbf{I} \otimes \mathbf{R}_{\mathbf{B}_1}^T \\ \vdots \\ \mathbf{R}_{\mathbf{A}_n} \otimes \mathbf{I} - \mathbf{I} \otimes \mathbf{R}_{\mathbf{B}_n}^T \end{pmatrix}}_{\mathbf{L}} \mathrm{vec}\left(\mathbf{R_X}\right) = 0. \tag{2}$$

Here the Kronecker product

$$\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} a_{1,1}B & \cdots & a_{1,n}B \\ \vdots & \ddots & \vdots \\ a_{m,1}B & \cdots & a_{m,n}B \end{pmatrix},$$

where $a_{i,j}$ is the (i,j)-th element of $\mathbf{A}$, and $\mathrm{vec}(\mathbf{A})$ vectorizes a matrix $\mathbf{A}$ column-wise. Liang et al. solve system (2) by

1. Calculating the eigenvector $\mathbf{y}$ corresponding to the smallest eigenvalue of $\mathbf{L}$
2. Forming $\mathbf{Y} = \mathrm{vec}^{-1}(\mathbf{y})$
3. Setting $\mathbf{R_X} = |\mathbf{U}\mathbf{V}^T|$ where the singular value decomposition of $\mathbf{Y} = \mathbf{U}\mathbf{S}\mathbf{V}^T$

Here,

$$|\mathbf{A}| = \begin{cases} \mathbf{A} & \text{if } \det(\mathbf{A}) \geq 0 \\ -\mathbf{A} & \text{if } \det(\mathbf{A}) < 0. \end{cases}$$

For all these separable solutions, errors in the calculation of the optimal rotation $\mathbf{R_X}$ get carried into the calculations of the optimal translation $\mathbf{t_X}$. In order to minimize these errors, simultaneous solutions for $\mathbf{AX} = \mathbf{XB}$ were created. However, these solutions have their own problems as will be discussed.

2.2 Simultaneous Solutions for AX=XB

Chen in [3] believes that separating the orientational component from the positional component, which implies that one has nothing to do with the other, is invalid. Thus, Chen creates a new solution, based on screw theory, that simultaneously solves the orientational component with the positional component. Specifically, he finds that the $\mathbf{AX} = \mathbf{XB}$ problem can be reduced to an absolute orientation problem of finding the best rigid transformation ($\mathbf{R_X}$ and $\mathbf{t_X}$) that transforms the camera screw axis to the robot screw axis.

Daniilidis and Bayro-Corrochano describe an algebraic interpretation of Chen's screw theory method via dual quaternions in [6, 7]. Specifically, they use the vector portions from the dual-quaternion representations $\mathbf{a}_i + \mathbf{a}'_i$ and $\mathbf{b}_i + \mathbf{b}'_i$ of $\mathbf{A}_i$ and $\mathbf{B}_i$ respectively to create the matrix

$$\begin{aligned} \mathbf{T} &= \begin{pmatrix} \mathbf{S}_1^T & \mathbf{S}_2^T & \dots & \mathbf{S}_n^T \end{pmatrix}^T \\ \mathbf{S}_i &= \begin{pmatrix} \overrightarrow{\mathbf{a}_i} - \overrightarrow{\mathbf{b}_i} & \mathrm{Sk}\left(\overrightarrow{\mathbf{a}_i} + \overrightarrow{\mathbf{b}_i}\right) & 0 & 0 \\ \overrightarrow{\mathbf{a}'_i} - \overrightarrow{\mathbf{b}'_i} & \mathrm{Sk}\left(\overrightarrow{\mathbf{a}'_i} + \overrightarrow{\mathbf{b}'_i}\right) & \overrightarrow{\mathbf{a}_i} - \overrightarrow{\mathbf{b}_i} & \mathrm{Sk}\left(\overrightarrow{\mathbf{a}_i} + \overrightarrow{\mathbf{b}_i}\right) \end{pmatrix} \end{aligned}$$

Using the singular value decomposition on $\mathbf{T}$, Daniilidis and Bayro-Corrochano show that the dual-quaternion representation for the unknown $\mathbf{X}$ can be calculated as a linear combination of the last two right singular vectors of $\mathbf{T}$. It should be noted that the authors developed a similar method through the use of Clifford Algebra in [2]. Zhao and Liu also develop a similar method through the algebraic properties of screw theory in [27].

Lu and Chou [15] apply the quaternions via the eight step method to solve the robot-sensor calibration problem simultaneously. Specifically, by the use of quaternions, they can simplify the problem to a single linear system which they solve using Gaussian elimination and Schur decomposition.

Andreff et al. are the first to apply the Kronecker product to simultaneously solve the robot-sensor problem in [1]. They reformulate the robot-sensor problem into a linear system of the form

$$\begin{pmatrix} \mathbf{I} - \mathbf{R}_{\mathbf{B}_i} \otimes \mathbf{R}_{\mathbf{A}_i} & 0 \\ \mathbf{t}_{\mathbf{B}_i}^T \otimes \mathbf{I} & \mathbf{I} - \mathbf{R}_{\mathbf{A}_i} \end{pmatrix} \begin{pmatrix} \mathrm{vec}(\mathbf{R}_\mathbf{X}) \\ \mathbf{t}_\mathbf{X} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{t}_{\mathbf{A}_i} \end{pmatrix}.$$

Andreff et al. prove that at least two independent general motions with non-parallel axes are needed to have a unique solution to the linear system. A problem with this method is that due to noise the solution for $\mathbf{R}_\mathbf{X}$ may not necessarily be an orthogonal matrix. Thus, an orthogonalization step for the orientational component has to be taken. However, the corresponding positional component is not recalculated, which causes errors in the solution. Therefore, Andreff et al. suggest separating the orientational and positional components as was shown in the work of Liang et al. (see Section 2.1) in [14].

2.3 Iterative Solutions for AX=XB

Simultaneous solutions were developed to solve the problem of orientational errors propagating into the positional errors. Another option to solve this problem is to create an iterative solution for $\mathbf{AX} = \mathbf{XB}$. Zhuang and Shiu propose a one-step iterative method, based on minimizing $\|\mathbf{AX} - \mathbf{XB}\|$ with the Levenberg-Marquardt algorithm in [30]. The iterative method solves both the orientational and positional components simultaneously. Furthermore, the method is not dependent on robot orientation $\mathbf{R}_{\mathbf{B}_i}$ information. Fassi and Legnani propose a similar algorithm in [9]. This paper also provides a geometric interpretation of the hand-eye calibration problem. Wei et al. [25] create an efficient iterative method that is optimized by the sparse structure of the corresponding normal equations.

Horaud and Dornaika in [11] also propose to solve the orientational and positional components simultaneously using an iterative method. However, their method is based on using the quaternion representation for the orientational component.

Mao et al. [16] apply the Kronecker product in their iterative formulation. An issue with the Mao et al. optimization problem is that the solution is based on the initial condition. Therefore, different initial conditions could result in varying solutions. A remedy to this problem is to use convex optimization as shown in the work of Zhao [26]. Zhao claims that his Kronecker product algorithm is very fast and not dependent on an initial condition. However, their setup gives no guarantee that the orientational component $\mathbf{R}_\mathbf{X}$ of the solution is a rotation matrix. Therefore, his algorithm may cause errors that are similar to the errors of Andreff et al. [1]. Shi et al. [20] have a similar formulation to Zhao (thus similar problems), but their iterative algorithm optimizes motion selection to improve accuracy and to avoid degenerate cases.

Strobl and Hirzinger create an iterative method that is based on a parameterization of a stochastic model in [22]. This iterative method is novel since it creates an inherent algorithm to weight the orientational and positional components to optimize the accuracy of the method. Kim et al. extend this formulation in [12] with the use of the Minimum Variance method.

These iterative methods get rid of the propagation of orientational errors into the positional component. However, solving the robot-sensor calibration method in this manner can be computationally taxing since these methods often contain complex optimization routines. In addition, as the number of equations (n) gets larger, the differences between iterative solutions and closed-form solutions often get smaller. Thus, one has to decide whether the accuracy of an iterative solution is worth the computational costs.

3. AX=YB SOLUTIONS

In this section we will give an overview of techniques to solve $\mathbf{AX} = \mathbf{YB}$. The methods for solving this system are very similar to the $\mathbf{AX} = \mathbf{XB}$ problems, i.e., the methods can be organized into three groups: separable solutions, simultaneous solutions, and iterative solutions.

Wang proposes the $\mathbf{AX} = \mathbf{YB}$ problem in [24], though he assumes that one of the unknowns is given. Zhuang et al. were the first to give a separable closed-form solution via quaternions in [29]. Dornaika and Horaud extend Zhuang et al.'s separable solution to give a more accurate separable closed-form solution via quaternions in [8]. Shah creates a formulation based on Kronecker product in [19].

Li et al. look at simultaneous closed-form solutions via dual-quaternions and Kronecker products in [13]. Their formulations follow the methodology of the $\mathbf{AX} = \mathbf{XB}$ formulation of dual quaternions of Daniildis [7] and the formulation of Kronecker product of Andreff et al. [1].

Iterative solutions for the $\mathbf{AX} = \mathbf{YB}$ problem were first introduced in the work of Remy et al. [18]. Here they define a nonlinear optimization problem and use the Levenberg-Marquardt method to solve it. Hirsh et al. develop an iterative method in [10] that optimizes the orientational and positional components separately, while Strobl and Hirzinger create an iterative method [22] that simultaneously solves the orientational and positional components. Their method is based on a parameterization of a stochastic model which is identical to their $\mathbf{AX} = \mathbf{XB}$ model. Kim et al. also use a model [12] identical to their $\mathbf{AX} = \mathbf{XB}$ model to simultaneously solve $\mathbf{AX} = \mathbf{YB}$ using the Minimum Variance method.

4. CONCLUSION

In this paper, we give an overview of methods to solve the robot-sensor calibration problem of the forms $\mathbf{AX} = \mathbf{XB}$ and $\mathbf{AX} = \mathbf{YB}$ for the evaluation of perception systems. Each form's solutions can be split into three categories: separable solutions, simultaneous solutions, and iterative solutions. The separable solutions are simple and fast solutions; however, errors calculated from the orientational component get carried over to the positional component. As a result, simultaneous solutions were developed. However, these solutions produce variable results depending on the scaling of the positional component. To weight the orientational and positional components, iterative methods were created. However, though these solutions are often more accurate, the solutions are often complex and generally depend on starting criteria. In addition, there is generally no guarantee that the convergent solution is the optimal solution. Thus, users must decide which type of method to use for evaluation which is dependent on their desired accuracy and complexity.

5. REFERENCES

[1] N. Andreff, R. Horaud, B. Espiau On-line Hand-Eye Calibration. In *Second International Conference on 3-D Digital Imaging and Modeling (3DIM'99)*, pages 430–436, 1999.

[2] E. Bayro-Corrochano, K. Daniilidis, G. Sommer Motor Algebra for 3D Kinematics: The Case of the Hand-Eye Calibration. In *Journal of Mathematical Imaging and Vision*, 13: 79–100, 2000.

[3] H. H. Chen A screw motion approach to uniqueness analysis of head-eye geometry. In *IEEE Proceedings of Computer Vision and Pattern Recognition (CVPR'91)*, pages 145–151, 1991.

[4] J. C. K. Chou, M. Kamel Quaternions approach to solve the kinematic equation of rotation, $A_a A_x = A_x A_b$, of a sensor-mounted robotic manipulator. In *IEEE International Conference on Robotics and Automation*, pages 656–662, 1988.

[5] J. C. K. Chou, M. Kamel Finding the Position and Orientation of a Sensor on a Robot Manipulator Using Quaternions. In *The International Journal of Robotics Research*, 10(3): 240–254, 1991.

[6] K. Daniilidis, E. Bayro-Corrochano The dual quaternion approach to hand-eye calibration. In *Proceedings of the 13th International Conference on Pattern Recognition*, pages 318–322, 1996.

[7] K. Daniilidis Hand-Eye Calibration Using Dual Quaternions. In *The International Journal of Robotics Research*,18(3): 286 – 298, 1999.

[8] F. Dornaika, R. Horaud, Simultaneous robot-world and hand-eye calibration, IEEE Transactions on Robotics and Automation, 14(4): 617 – 622, 1998.

[9] I. Fassi, G. Legnani Hand to Sensor Calibration: A Geometrical Interpretation of the Matrix Equation AX=XB. In *Journal of Robotic Systems*, 22(9): 497 – 506, 2005.

[10] R. L. Hirsh, G. N. DeSouza, A. C. Kak An Iterative Approach to the Hand-Eye and Base-World Calibration Problem. In *Proceedings of the 2001 IEEE International Conference on Robotics and Automation*, 3: 2171 – 2176, 2001.

[11] R. Horaud, F. Dornaika Hand-Eye Calibration In *International Journal of Robotics Research*, 14(3): 195 –210, 1995.

[12] S. Kim, M. Jeong, J. Lee, J. Lee, K. Kim, B. You, S. Oh Robot Head-Eye Calibration Using the Minimum Variance Method. In *Proceedings of the 2010 IEEE International Conference on Robotics and Biomimetics*, pages 1446 – 1451, 2010.

[13] A. Li, L. Wang, D. Wu Simultaneous robot-world and hand-eye calibration using dual-quaternions and kronecker product. In*International Journal of the Physical Sciences*, 5(10): 1530 –1536, 1995.

[14] R. Liang, J. Mao Hand-Eye Calibration with a New Linear Decomposition Algorithm. In *Journal of Zhejiang University*, 9(10):1363–1368, 2008.

[15] Y. Lu, J. C. K. Chou Eight-Space Quaternion Approach for Robotic Hand-eye Calibration. In *Systems, IEEE International Conference on Man and Cybernetics*, 4: 3316 – 3321, 1995.

[16] J. Mao, X. Huang, L. Jiang A Flexible Solution to AX=XB for Robot Hand-Eye Calibration. In *Proceedings of the 10th WSEAS International Conference on Robotics, Control and Manufacturing Technology*, pages 118–122, 2010.

[17] F. Park, B. Martin Robot Sensor Calibration: Solving AX = XB on the Euclidean Group. In *IEEE Transactions on Robotics and Automation*, 10(5): 717–721, 1994.

[18] S. Remy, M. Dhome, J. M. Lavest, N. Daucher Hand-eye Calibration. In *Proceedings of the 1997 IEEE/RSJ International Conference on Intelligent Robots and Systems*, 2:1057 – 1065, 1997.

[19] M. I. Shah. Solving the Robot-World/Hand-Eye Calibration Problem using the Kronecker Product. Technical report, Department of Mathematics and Statistics, Loyola University in Maryland, 2012.

[20] F. Shi, J. Wang, Y. Liu An Approach to Improve Online Hand-Eye Calibration. In *Pattern Recognition and Image Analysis*, 3522: 539 – 567, 2005.

[21] Y. Shiu, S. Ahmad Calibration of Wrist-Mounted Robotic Sensors by Solving Homogeneous Transform Equations of the Form AX = XB. In *IEEE Transactions on Robotics and Automation*, 5(1):16–29, 1989.

[22] K. H. Strobl, G. Hirzinger Optimal Hand-Eye Calibration. In *Proceedings of the 2006 IEEE/RSJ International Conference on Intelligent Robots and Systems*, pages 4647 – 4653, 2006.

[23] R. Tsai, R. Lenz A New Technique for Fully Autonomous and Efficient 3D Robotics Hand/Eye Calibration. In *IEEE Transactions on Robotics and Automation*, 5(3):345-358, 1989.

[24] C. Wang Extrinsic Calibration of a Vision Sensor Mounted on a Robot. In *IEEE Transactions on Robotics and Automation*, 8(2): 161–175, 1992.

[25] G. Wei, K. Arbter, G. Hirzinger Active self-calibration of robotic eyes and hand-eye relationships with model identification. In *IEEE Transactions on Robotics and Automation*, 14(1): 158 – 166, 1998.

[26] Z. Zhao Hand-Eye Calibration Using Convex Optimization. In *2011 IEEE International Conference on Robotics and Automation*, pages 2947–2952, 2011.

[27] Z. Zhao, Y. Liu Hand-Eye Calibration Based on Screw

Motions. In *18th International Conference on Pattern Recognition(ICPR'06)*, pages 1022 – 1026, 2006.

[28] H. Zhuang, Z. S. Roth Comments on "Calibration of Wrist-Mounted Robotic Sensors by Solving Homogeneous Transform Equations of the Form AX = XB." In *IEEE Transactions on Robotics and Automation*, 7(6): 877–878, 1991.

[29] H. Zhuang, Z. S. Roth, R. Sudhakar Simultaneous Robot/World and Tool/Flange Calibration by Solving Homogeneous Transformation Equations of the form AX=YB. In *IEEE Transactions on Robotics and Automation*, 10(4):549 – 554, 1994.

[30] H. Zhuang, Y . C. Shiu A Noise-Tolerant Algorithm for Robotic Hand-Eye Calibration with or without Sensor Orientation Measurement. In *IEEE Transactions on Systems, Man, and Cybernetics*, 23(4): 1168 – 1175, 1993.