The LCM-lattice: a crossroads for combinatorics, topology and commutative algebra

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Setting

Suppose that $\mathbb{K}$ is a field, let

$$S = \mathbb{K}[x_1, \cdots, x_d],$$

and consider only those ideals in S generated by monomials.

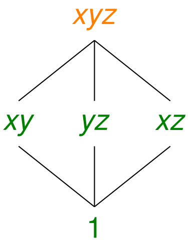
Running examples

- $M = (xy, yz, xz)$ in $S = \mathbb{K}[x, y, z]$
- $N = (ab, bc, cd, de, ef)$ in $S = \mathbb{K}[a, b, c, d, e, f]$

The LCM-lattice

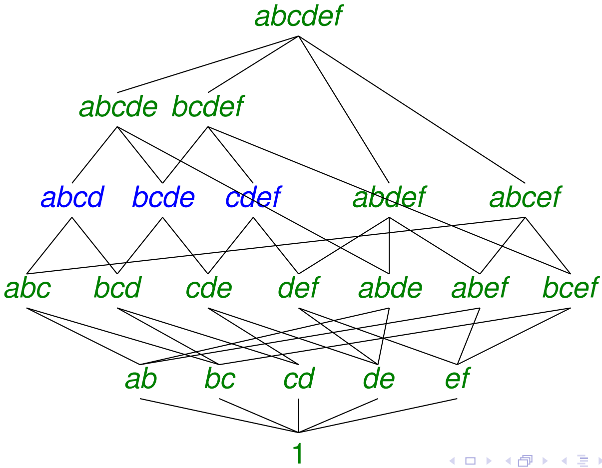
Definition: The ***LCM-lattice*** is the set L_I of least common multiples of the monomial generators of I, ordered by divisibility.

Example: For $M = (xy, yz, xz)$ the Hasse diagram of L_M is



The LCM-lattice

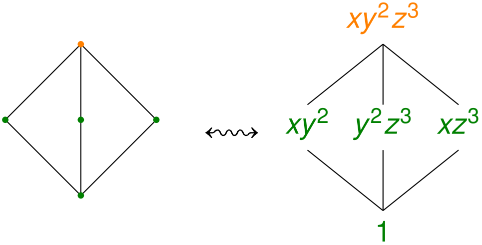
Example: The LCM-lattice L_N has Hasse diagram



Atomic lattices and monomial ideals

Theorem [Phan, 2006] Every finite atomic lattice L may be associated to a monomial ideal $I(L)$. Furthermore, the LCM-lattice of $I(L)$ is isomorphic to L.

Process: For every finite atomic lattice, find a monomial labeling of the lattice elements so that LCMs are respected.



A commutative algebra problem

Open Problem [Kaplansky, 1960s]

Non-iteratively construct the ***minimal free resolution*** of the S-module S/I using the combinatorial data of the generators of I and the arithmetic in the field $\mathbb{K}$.

A minimal resolution of S/M

Example: For the ideal $M = (xy, yz, xz)$, the module S/M has the following minimal free resolution.

$$S \xleftarrow[\partial_1]{[\, xy \;\; yz \;\; xz \,]} S^3 \xleftarrow[\partial_2]{\begin{bmatrix} z & 0 \\ -x & x \\ 0 & -y \end{bmatrix}} S^2 \leftarrow 0$$

Note: Among others, the monomial relation $z \cdot (xy) - x \cdot (yz) = 0$ is encoded in this resolution.

What is a resolution? (Linear Algebra Version)

Let A be a $p \times q$ matrix whose entries are from $\mathbb{K}$.

Problem: Solve the system of linear equations $AX = 0$.

Methodology: Consider A as a linear transformation and write X for a matrix of column vector solutions $X_1, X_2, \cdots, X_n$.

We may therefore encode solutions in the sequence

$$\mathbb{K}^p \xleftarrow{A} \mathbb{K}^q \xleftarrow{X} \mathbb{K}^n.$$

What is a resolution? (Linear Algebra Version)

Our set of solutions is complete if and only if $n \geqslant q - \text{rank}(A)$.

If the solutions are linearly independent, i.e. if $n = q - \text{rank}(A)$ then we have an *exact* sequence

$$\mathbb{K}^p \xleftarrow{A} \mathbb{K}^q \xleftarrow{X} \mathbb{K}^q \leftarrow 0.$$

We call this sequence a **free resolution** of

$$\text{coker}(A) = \mathbb{K}^p/\text{Im}(A).$$

What is a resolution? (Commutative Algebra Version)

Let A be a $p \times q$ matrix with entries in the polynomial ring S.

Problem: Solve the system of S-linear equations $AX = 0$.

Methodology: Consider A as a map between free modules

$$S^p \xleftarrow{A} S^q$$

so that

- the solutions we seek are elements of $\ker(A)$,
- a complete set of generators for $\ker(A)$ gives a complete set of solutions.

What is a resolution? (Commutative Algebra Version)

Snag: It may be the case that ***any*** complete set of generators for $\ker(A)$ has relations between its elements.

Workaround: Suppose that there are n generators for $\ker(A)$ and consider

$$S^p \xleftarrow{A} S^q \xleftarrow{X} S^n,$$

where a basis element of S^n is sent to a generator of $\ker(A)$.

We have encoded the relations between the generators of $\ker(A)$ in the S-module homomorphism X.

What is a resolution? (Commutative Algebra Version)

If we iterate this process and take care to choose the *smallest* number of generators necessary at each stage we produce a ***minimal free resolution*** of $\text{coker}(A) = S^p/\text{Im}(A)$.

$$S^p \xleftarrow{A} S^q \xleftarrow{X} S^n \cdots \xleftarrow{D} S^r \leftarrow 0$$

Terminology: The ranks $(p, q, n, \cdots, r)$ of the free modules in a minimal free resolution are called the ***Betti numbers*** of the S-module $\text{coker}(A)$.

Minimal resolutions of S/M

Example: For the ideal $M = (xy, yz, xz)$, the module S/M has the following isomorphic versions of its minimal resolution.

$$S \xleftarrow[\partial_1]{\begin{bmatrix} xy & yz & xz \end{bmatrix}} S^3 \xleftarrow[\partial_2]{\begin{bmatrix} z & 0 \\ -x & x \\ 0 & -y \end{bmatrix}} S^2 \leftarrow 0$$

and

$$S \xleftarrow[\partial_1]{\begin{bmatrix} xy & yz & xz \end{bmatrix}} S^3 \xleftarrow[\partial_2']{\begin{bmatrix} z & z \\ -x & 0 \\ 0 & -y \end{bmatrix}} S^2 \leftarrow 0$$

Open Problem (reformulated): Read the data of minimal resolutions from a combinatorial object.

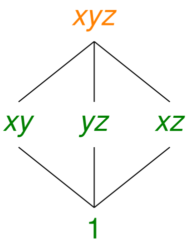
Homology of intervals in the LCM-lattice

Theorem [Gasharov, Peeva, and Welker, 1998]
Let I be a monomial ideal in S. The Betti numbers of S/I are given by the $\mathbb{K}$-vectorspace dimensions of the homology groups of spaces associated to open intervals in the LCM-lattice.

Definition (intuitive)
Homology is an algebraic measurement of the holes that exist in a topological space. For our purposes, homology is calculated by encoding topological data into an exact sequence of vector spaces over $\mathbb{K}$.

An LCM-lattice which supports minimal resolutions

The LCM-lattice of $M = (xy, yz, xz)$



The Betti numbers for R/M are $(1, 3, 2)$.

Creating a resolution from a poset

Construction:

			Homology		Monomial data	
L_I	$\rightsquigarrow$	$\{\Delta_m\}_{m \in L_I}$	$\rightsquigarrow$	$\mathcal{D}(L_I)$	$\rightsquigarrow$	$\mathcal{F}_I$
LCM lattice		Topological Spaces		Sequence of Vector Spaces		Sequence of Free Modules

Theorem [C, 2010]: The sequence $\mathcal{F}_I$ is the minimal free resolution for a wide class of monomial ideals.

Goal: For ideals whose LCM-lattice does not submit to this process, apply the construction to another appropriate poset.

Rigidity (with S. Mapes)

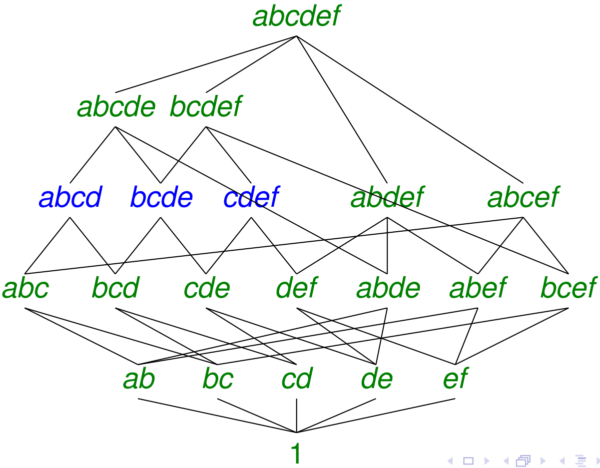
Definition [Miller and Peeva]: A monomial ideal is ***rigid*** if there exists a unique choice of S-basis for every free module in its minimal resolution. Furthermore, such a choice results in a unique representation for the monomial maps in the resolution.

Theorem [C and Mapes]: Rigidity is detectable using properties of the LCM-lattice.

Non-Example: The ideal $M = (xy, yz, xz)$ is not rigid, since we found two versions of its minimal free resolution.

A rigid LCM-lattice

Example: The LCM-lattice of *N* has Hasse diagram



$N = (ab, bc, cd, de, ef)$ is rigid

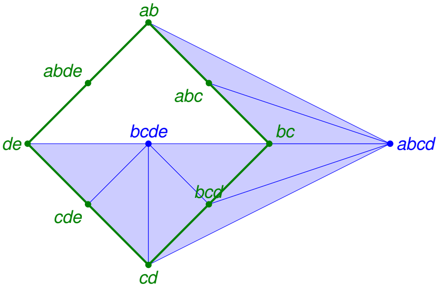
The minimal resolution of S/N is

$$S \xleftarrow[\partial_1]{\begin{bmatrix} ab & bc & cd & de & ef \end{bmatrix}} S^5 \xleftarrow[\partial_2]{\begin{bmatrix} -c & 0 & 0 & 0 & -de & -ef & 0 \\ a & -d & 0 & 0 & 0 & 0 & -ef \\ 0 & b & -e & 0 & 0 & 0 & 0 \\ 0 & 0 & c & -f & ab & 0 & 0 \\ 0 & 0 & 0 & d & 0 & ab & bc \end{bmatrix}} S^7 \xleftarrow[\partial_3]{\begin{bmatrix} -de & ef & 0 & 0 \\ -ae & 0 & 0 & -ef \\ -ab & 0 & 0 & -bf \\ 0 & 0 & -ab & -bc \\ c & 0 & -f & 0 \\ 0 & -c & d & 0 \\ 0 & a & 0 & d \end{bmatrix}} S^5 \xleftarrow[\partial_4]{\begin{bmatrix} -f \\ -d \\ -c \\ a \end{bmatrix}} S \leftarrow 0.$$

The Betti numbers of S/N are $(1, 5, 7, 4, 1)$.

A space containing too much information

Example: For the ideal N, data from the monomial *abcde* in the LCM-lattice L_N produces the space Δ_{abcde}.



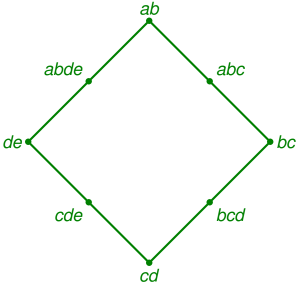
Pruning a poset

Process for pruning
Prune a monomial m from the LCM-lattice when the associated space Δ_m has no homology. Write $\mathcal{P}(L_I)$ for the poset consisting only of monomials which remain after pruning.

Lemma [C and Mapes]
Let I be a monomial ideal. Then there exists an isomorphism in homology between the open intervals in the LCM-lattice L_I and the open intervals in the pruned LCM-lattice $\mathcal{P}(L_I)$.

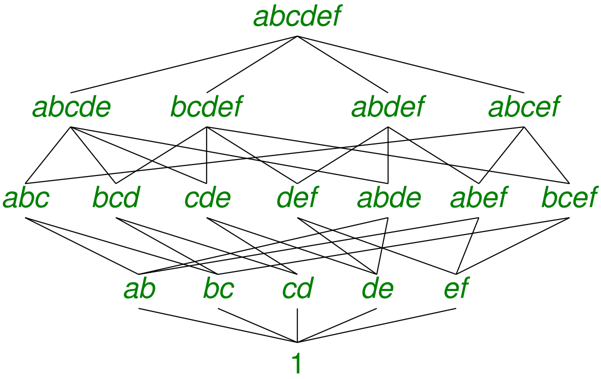
Pruning a poset

Example: For the ideal N, data from the monomial *abcde* in the pruned LCM-lattice $\mathcal{P}(L_N)$ produces the space Δ_{abcde}.



Pruning our rigid example

Example: The *pruned* LCM-lattice $\mathcal{P}(L_N)$ has Hasse diagram



Main result

Theorem [C and Mapes]
The minimal free resolution of a rigid monomial ideal I is supported on the pruned LCM-lattice $\mathcal{P}(L_I)$.

What questions can resolutions answer?

Do 7 sufficiently general points in $\mathbb{P}^3$ lie on a cubic?