Research Summary and Current Projects

Timothy B.P. Clark

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Abstract

I conduct research in the area of Commutative Algebra and use combinatorial and topological methods to describe algebraic objects. I am interested in using partially ordered sets and other combinatorial structures to explicitly describe Betti numbers and free resolutions of modules over a polynomial ring whose coefficients are from a field. This document contains relevant background material, key results from the papers that I have written, questions that have arisen throughout the research process, and progress made in ongoing research.

Introduction

Let $R = \Bbbk[x_1, \ldots, x_n]$ be a polynomial ring over a field $\Bbbk$. We consider R with its natural $\mathbb{N}^n$-grading (multigrading), so that $\deg(m) = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ for every monomial $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ of R. Throughout this document, we refer to a monomial and its multidegree interchangeably.

Definition 1. *Let M be a multigraded module of R. A* **multigraded free resolution** *of M is an exact sequence of free multigraded R-modules and multigraded morphisms,*

$$\mathcal{F} : \cdots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0$$

for which $\mathrm{Coker}(d_1) \cong M$. *The resolution $\mathcal{F}$ is said to be* **minimal** *if the rank of each free module F_i is minimal among all free resolutions of M.*

When $\mathcal{F}$ is minimal β_i, the rank of the module F_i, is called the ith Betti number of M. Further, write $\beta_{i,\alpha}$ for the multigraded Betti number in multidegree α and homological degree i. We therefore may decompose the module F_i as

$$F_i = \bigoplus_{\alpha \in \mathbb{Z}^n} F_{i,\alpha} = \bigoplus_{\alpha \in \mathbb{Z}^n} R(-\alpha)^{\beta_{i,\alpha}},$$

a direct sum of shifted free modules of total rank $\beta_i = \sum_{\alpha \in \mathbb{Z}^n} \beta_{i,\alpha}$.

Remark 2. *Every multigraded module M has a minimal free resolution, which is unique up to multigraded isomorphism.*

In the early 1960s, Kaplansky posed the specific problem of giving a nonrecursive construction for the minimal free resolution of the module R/N in the case where N is an ideal minimally generated by monomials. In the intervening years, free resolutions of monomial ideals have been extensively studied, and (minimal) free resolutions have been constructed in many specific cases ([2],[3],[15],[19],[24],[27],[29]), but even the general problem for monomial ideals remains open.

1 Poset resolutions and monomial ideals

Let $(P, <)$ be a finite partially ordered set with a minimum element and set of atoms A. In [9], we describe the process by which a complex of multigraded modules is constructed from P. This technique produces a sequence of vector spaces

$$\mathcal{D}_\bullet(P) : \cdots \longrightarrow \mathcal{D}_i \xrightarrow{\varphi_i} \mathcal{D}_{i-1} \longrightarrow \cdots \longrightarrow \mathcal{D}_1 \xrightarrow{\varphi_1} \mathcal{D}_0$$

which is a complex if P is a ranked poset. We use the information from a poset map $\eta : P \to \mathbb{N}^n$ as the data in the *homogenization* of $\mathcal{D}_\bullet(P)$ to produce $\mathcal{F}(\eta)$, an approximation to a free resolution of the monomial ideal whose generators have multidegrees which are the images (under η) of the atoms in P.

Definition 3. *If $\mathcal{F}(\eta)$ is an exact complex of multigraded modules, then we say that it is a* **poset resolution**.

Heuristically, a poset supports a resolution when it correctly identifies the rank and location of the free modules appearing in the resolution and simultaneously has covering relations which mirror the action of the differential in the resolution.

Thematic Question. *Which posets are capable of supporting the (minimal) free resolution of a multigraded module? Of a monomial ideal?*

2 Lattice-linear monomial ideals

In [16], Gasharov, Peeva and Welker introduce an important combinatorial object associated to a monomial ideal N. They define the *lcm-lattice*, the set L_N of least common multiples of the generators of N, with ordering given by $m < m' \in L_N$ if and only if m divides m'.

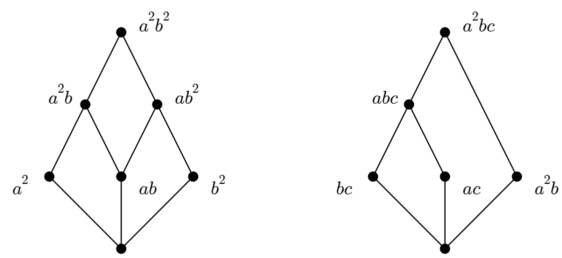
Furthermore, they prove that every multigraded Betti number $\beta_{i,m}$ of a monomial ideal can be calculated using the homology of the (order complex of) the open intervals of the form $(1, m)$ in the lcm-lattice. This makes the lcm-lattice a natural candidate for the poset construction above.

Definition 4. *Let $\mathcal{F}$ be a minimal multigraded free resolution of R/N. The ideal N is* **lattice-linear** *if the action of the differential of $\mathcal{F}$ may be read from the covering relations in the lcm-lattice L_N.*

The main result of [9] provides a necessary and sufficient condition for an ideal to be lattice-linear.

Theorem 5. *Let $\eta : L_N \longrightarrow \mathbb{N}^n$ be the poset map which sends a monomial to its multidegree. A monomial ideal N is lattice-linear if and only if $\mathcal{F}(\eta)$ is its minimal free resolution.*

Example 1. *The monomial ideal $M = (a^2, ab, b^2)$ is lattice-linear since the monomials a^2b and ab^2 are not comparable in L_M (pictured on the left), whereas the monomial ideal $N = (bc, ac, a^2b)$ is not lattice-linear since the monomials abc and a^2bc are comparable in L_N (pictured on the right).*



In addition to having a minimal free resolution which is constructible from the lcm-lattice, the class of lattice-linear ideals includes the ideals with linear resolutions, the Scarf ideals, and the monomial ideals which are generic in the sense of [2]. This leads naturally to the following questions.

Question. *What other classes of monomial ideals are subclasses of the class of lattice-linear ideals?*

Question. *What are the combinatorial properties of L_N which guarantee the lattice-linearity of N?*

3 Rigid ideals

In [11], Sonja Mapes and I explore the class of *rigid* monomial ideals, a generalization of a class of ideals first studied by Miller in [22]. In the three-variable case, Miller showed that the minimal resolution of such an ideal may be constructed using a rigid embedding of the multidegrees appearing in the resolution into $\mathbb{N}^3$. Rigid ideals include as subclasses the generic monomial ideals and the monomial ideals whose minimal resolution is supported on their Scarf complex.

By definition, a rigid ideal has the following two properties. Every nonzero multigraded Betti number equals one, and multigraded Betti numbers which are nonzero in the same homological degree correspond to incomparable monomials in the lcm-lattice. The monomial ideals N and M of Example 1 are rigid and non-rigid, respectively.

Definition 6. *Let $\mathcal{F}$ be a multigraded free resolution of a multigraded module. An* **automorphism** *of $\mathcal{F}$ is a collection of multigraded (degree 0) isomorphisms $f_i : F_i \to F_i$ which has the property that $d'_i \circ f_i = f_{i-1} \circ d_i$ for every $i \geq 1$. Here, $\{d_i\}$ and $\{d'_i\}$ are the representatives of the differential of $\mathcal{F}$ which come as a result of distinct basis choices.*

For an arbitrary monomial ideal, the isomorphism f_i may be realized as an element of $GL_{\beta_i}(R)$ for every $0 \leq i \leq p$ and as such, the automorphism group of $\mathcal{F}$ is a subgroup of $\bigoplus_{i=0}^{p} GL_{\beta_i}(R)$. Therefore, once the notion of a resolution automorphism is established, a natural question arises.

Question. *How can the theory of algebraic groups inform our understanding of resolution automorphisms and rigidity?*

We say that an automorphism of $\mathcal{F}$ is **trivial** if for any ordered choice of basis, the maps representing the isomorphisms $\{f_i\}$ are diagonal matrices with units of $\Bbbk$ along the diagonal. In our first result, we reprove an unpublished result of Miller and Peeva.

Proposition 7. *[11] If M is a monomial ideal then the automorphisms of the minimal resolution of R/M are trivial if and only if M is rigid.*

This result allows one to think of a rigid ideal's minimal resolution as essentially unique, in that there is a unique choice of multigraded bases for said resolution. Our interest in studying rigid ideals stems from a desire to identify posets which encode the data of both the modules and the maps in a minimal free resolution of a monomial ideal. The techniques we develop allow us to prove that for a subclass of rigid monomial ideals, the lcm-lattice supports the minimal resolution. Precisely, we focus attention on the subclass of **concentrated** rigid ideals, which have all non-contributing multidegrees appearing higher in the lcm-lattice than the multidegrees which do contribute. The ideal M from Example 1 is a rigid, concentrated ideal.

Theorem 8. *[11] A rigid monomial ideal is concentrated if and only if it is lattice-linear.*

In ongoing work [12], we extend Theorem 8 to the subposet of L_N which consists of contributing multidegrees, and are able to give a construction for the minimal free resolution of any rigid ideal.

Theorem 9. *[12] A rigid monomial ideal is minimally resolved on its poset of contributing multidegrees.*

Although this method allows the construction of a minimal poset resolution, we can currently detect rigidity only by examining the lcm-lattice. As such, answering the following question is an area of ongoing research.

Question. *Are combinatorial criteria for rigidity encoded in the monomial generators of an ideal?*

4 Lcm-lattices and resolution automorphisms

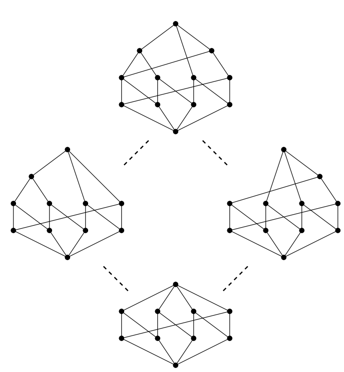
We now discuss rigid ideals from another perspective. The method developed by Mapes [20] for comparing lcm-lattices can be used to construct the minimal free resolution of certain non-concentrated rigid ideals using the minimal free resolution of an appropriately comparable concentrated rigid ideal.

Let $\mathcal{L}(n)$ be the set of all finite atomic lattices with n ordered atoms. Phan in [25] defines a partial order on $\mathcal{L}(n)$ by $P \geq Q$ if there exists a join preserving map $f : P \to Q$ which is a bijection on atoms. Further, he shows that under this partial order, $\mathcal{L}(n)$ is itself a finite atomic lattice. Moreover, a result of Gasharov, Peeva and Welker [16] indicates that total Betti numbers weakly increase as one moves up chains in $\mathcal{L}(n)$. Since every finite atomic lattice is the lcm-lattice of some monommial ideal, $\mathcal{L}(n)$ can be thought of as the lattice of all monomial ideals with n ordered generators up to equivalence of lcm-lattices.

Let $\beta_i = \sum \beta_{i,m}$ be the total Betti numbers of the ideal M and write $\beta = (\beta_0, \beta_1, \ldots, \beta_t)$ for the sequence of total Betti numbers. It is reasonable to fix subposets of $\mathcal{L}(n)$ which consist of all the finite atomic lattices with the same total Betti numbers. We refer to these subposets as *Betti stratum* and denote them $\mathcal{L}(n)_\beta$. Given a rigid monomial ideal M whose sequence of Betti numbers is β we obtain results on the relationship between M and ideals whose lcm-lattices are in $\mathcal{L}(n)_\beta$.

Theorem 10. *[11] Let $P, Q \in \mathcal{L}(n)_\beta$ for some β. If P is rigid and $Q > P$ then Q is rigid. Furthermore, the minimal resolution of P is isomorphic to the minimal resolution of Q.*

Example 2. *Three dispersed rigid ideals which sit above a concentrated rigid ideal in the stratum $\beta = (1, 4, 4, 1)$ of $\mathcal{L}(4)$.*



Our results regarding rigid ideals also motivate the following question.

Question. *How can the technique of transferring resolution information upward in $\mathcal{L}(n)$ be applied to other classes of monomial ideals?*

5 Stable ideals

In addition to using the lcm-lattice and Betti subposet, I have conducted research on other posets which support minimal free resolutions. A monomial ideal N is *stable* if for every monomial $m \in N$, the monomial $m \cdot x_i/x_d \in N$ for every $1 \leq i < d$, where $d = \max(m)$ is the maximum index over all variables dividing m. Eliahou and Kervaire in [15] construct the minimal free resolution of a stable monomial ideal. While some stable ideals are lattice-linear, there exist even Borel-fixed ideals (a more restricted class) which are not. Therefore in [10], I interpret the minimal free resolution of a stable ideal using the theory of poset resolutions. For our purposes, the partially ordered under consideration is the set of so-called *Eliahou-Kervaire admissible symbols* on the monomial generators of N.

$$P_N = \left\{ (I, m) \mid I \subseteq \{1, \ldots, n-1\}, m \text{ a generator of } N, \max(I) < \max(m) \right\} \cup \{(\varnothing, 1)\}.$$

To provide the appropriate connection to poset resolutions, we define the poset map $\eta : P_N \longrightarrow \mathbb{N}^n$ as $(I, m) \mapsto \deg(x_I m)$. The main result in [10] is

Theorem 11. *[10] The complex $\mathcal{F}(\eta)$ is a minimal poset resolution of the stable ideal N.*

In order to establish this result, I show that the poset P_N is a CW-poset in the sense of Björner [6]. We may therefore view P_N as the face poset of a regular CW complex. This allows a topological interpretation of the resolution $\mathcal{F}(\eta)$ as a cellular resolution, recovering a result proven independently by Batzies and Welker [1] and Mermin [21].

Theorem 12. *[10] Suppose that N is a stable monomial ideal. Then the minimal free resolution $\mathcal{F}(\eta)$ is a minimal cellular resolution of R/N.*

6 Cellular resolutions of monomial ideals

A CW-complex X and its cell poset $P(X)$ may share certain topological properties. Recall that a CW-complex is said to be regular if the attaching maps are homeomorphisms. In the case when X is a regular CW-complex, X is CW-isomorphic to $P(X)$. In joint work with Alexandre Tchernev [13], we show the following.

Theorem 13. *[13] If N is a monomial ideal whose minimal free resolution $\mathcal{F}$ is supported on a regular CW complex, then $\mathcal{F}$ is a poset resolution.*

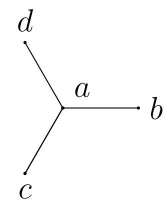
It is natural to consider an even wider class of CW complexes, since X and $P(X)$ may share topological properties even if they are not homeomorphic. Along with Alex Tchernev and Marco Varisco, I am working towards an answer to the following question.

Question. *Do monomial ideals whose (minimal) free resolution is supported on an arbitrary CW complex also admit a (minimal) poset resolution?*

7 Edge ideals and atomic lattices

Let G be a finite, simple graph with vertex set V and edge set E. The *edge ideal* I_G is the monomial ideal in $\Bbbk[V]$ whose minimal generators are in one-to-one correspondence with the edges of G. An edge ideal is therefore a squarefree monomial ideal with quadratic generators. Edge ideals were first studied by Villarreal [30] and have been the focus of numerous research programs in combinatorial commutative algebra.

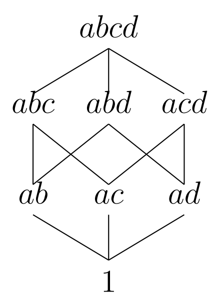
Example 3. *The following graph is a star with three edges and the associated edge ideal is minimally generated by the three monomials ab, ac, ad.*



I supervised an undergraduate independent study research course for Andrea Heyman. In her work, she [18] characterized the structure of the lcm-lattice for a subclass of edge ideals. Recall that a connected graph G on n vertices is a star if it is a complete bipartite graph with two vertex classes; one class which contains a single vertex and the other class which contains the remaining $n-1$ vertices. Heyman calls a graph G a **galaxy** if it is a disjoint union of stars. Using combinatorial techniques, she proves the following.

Theorem 14. *[18] The lcm-lattice of an edge ideal I_G is a Boolean lattice if and only if G is a galaxy.*

Example 4. *The lcm-lattice of the star edge ideal (ab, ac, ad), is a Boolean lattice on three atoms.*



This project has laid the groundwork for future (undergraduate) research, which broadly falls under the following question.

Question. *What other classes of graphs give rise to edge ideals with interesting lcm-lattices?*

If a combinatorial description for the lcm-lattice of an edge ideal can be established, one may then ask questions about the structure of the minimal free resolution of said edge ideal.

Goal. *Give an explicit construction for the minimal free resolution of an arbitrary edge ideal.*

Heyman's result therefore has an immediate consequence for the minimal resolution of an edge ideal which arises from a galaxy.

Theorem 15. *If G is a galaxy, then the ideal I_G is lattice-linear.*

8 The reduced lattice of T-flats of a matroid

My work with posets has also opened up a line of research related to resolutions of multigraded modules. More precisely, there is a combinatorial connection between poset resolutions and the lattice of T-flats. Along with Alex Tchernev and Amanda Beecher, we apply poset resolution techniques to the T-resolution of multigraded modules constructed by Tchernev in [28]. The modules in this resolution have a combinatorial description due to Beecher [4] using the homology of broken circuit complexes. In [5], we show that the maps of [28] can be reinterpreted combinatorially (and topologically) to give a more concrete description of the T-resolution.

Theorem 16. *[5] Let $\phi\colon\ U \longrightarrow W$ be a presentation of a multigraded module L. Suppose the matroid associated to of ϕ (under a fixed ordering of the columns in the matrix) is uniform. Then the reduced lattice of T-flats of the matroid of ϕ supports a poset resolution of the module L.*

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