A weak solution and a multinumerics solution of the coupled Navier-Stokes and Darcy equations

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Abstract

This paper introduces and analyzes two models coupling of incompressible Navier-Stokes equations with the porous media flow equations. A numerical method that uses continuous finite elements in the incompressible flow region and discontinuous finite elements in the porous medium, is proposed. Existence and uniqueness results under small data condition of the numerical solution are proved. Optimal a priori error estimates are derived. Numerical examples comparing the two models are provided.

1 Introduction

There is an increasing interest in coupling incompressible flow and porous media flow. Applications of such complex phenomena can be found in geosciences (modeling of the interaction of rivers with groundwater) and in health sciences (modeling of blood flow and organs). In this work, we consider the coupling of the nonlinear Navier-Stokes equations with the Darcy equations. Non-homogeneous boundary conditions are imposed on the boundary of the porous medium. This generalizes the weak problem defined in [16] where homogeneous boundary conditions were assumed. We also propose a numerical scheme that couples the continuous finite element method with the Discontinuous Galerkin (DG) method. Because of legacy codes, multinumerics approaches are attractive. In addition, one can take advantage of the benefits of the different methods used in the subdomains. On one hand, classical finite elements are popular for computational fluid dynamics. On the other hand, the advantages of DG methods include the flexible use of mesh adaptivity and high order of approximation. The DG methods we consider here are called primal DG methods and they are variations of interior penalty methods. These methods encompass the non-symmetric interior penalty Galerkin method (NIPG) [25, 26, 20], the incomplete interior penalty Galerkin method (IIPG) [9] and the symmetric interior penalty Galerkin method (SIPG) [29, 2]. In [16], the coupled problem is approximated by totally discontinuous elements. In [10, 4], the coupling of Navier-Stokes with Darcy with homogeneous boundary conditions has been analyzed by using Steklov-Poincaré operators and by using continuous finite elements in a Robin-Robin domain decomposition approach.

Let Ω be a bounded domain in $\mathbb{R}^2$, that is subdivided into two disjoint subdomains Ω_1 and Ω_2. Let Γ_{12} denote the interface between the subdomains: $\Gamma_{12} = \partial\Omega_1 \cap \partial\Omega_2$. We assume that Γ_{12} is a polygonal line. The flow in Ω_1 is incompressible and characterized by the Navier-Stokes equations:

$$-\nabla \cdot (2\mu D(\boldsymbol{u}_1) - p_1 \boldsymbol{I}) + \boldsymbol{u}_1 \cdot \nabla \boldsymbol{u}_1 = \boldsymbol{f}_1, \quad \text{in } \Omega_1, \tag{1}$$

$$\nabla \cdot \boldsymbol{u}_1 = 0, \quad \text{in } \Omega_1, \tag{2}$$

$$\boldsymbol{u}_1 = 0, \quad \text{on } \partial\Omega_1 \backslash \Gamma_{12} = \Gamma_1. \tag{3}$$

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The fluid velocity and pressure in Ω_1 are denoted by $\boldsymbol{u}_1$ and p_1 respectively. The coefficient $\mu > 0$ is the fluid viscosity, the function $\boldsymbol{f}_1$ is an external force acting on the fluid, $\boldsymbol{I}$ is the identity matrix and the matrix $\boldsymbol{D}(\boldsymbol{u}_1)$ is the stress tensor:

$$\boldsymbol{D}(\boldsymbol{u}_1) = \frac{1}{2}(\nabla \boldsymbol{u}_1 + \nabla {\boldsymbol{u}_1}^T). \tag{4}$$

The flow in Ω_2 is of Darcy type. We assume that the boundary $\Gamma_2 = \partial\Omega_2 \setminus \Gamma_{12}$ is the union of two disjoint sets $\Gamma_{2\mathrm{D}}$ and $\Gamma_{2\mathrm{N}}$ on which Dirichlet and Neumann boundary conditions are imposed.

$$\begin{aligned} -\nabla \cdot \boldsymbol{K}\nabla p_2 &= f_2, \text{ in } \Omega_2, && (5)\\ -\boldsymbol{K}\nabla p_2 &= \boldsymbol{u}_2, \text{ in } \Omega_2, && (6)\\ p_2 &= g_{\mathrm{D}}, \text{ on } \Gamma_{2\mathrm{D}}, && (7)\\ \boldsymbol{K}\nabla p_2 \cdot \boldsymbol{n}_2 &= g_{\mathrm{N}}, \quad \text{on } \Gamma_{2\mathrm{N}}. && (8) \end{aligned}$$

Similarly, the fluid velocity and pressure in Ω_2 are denoted by $\boldsymbol{u}_2$ and p_2 respectively. The function f_2 is an external force acting on the fluid, the functions g_{D} and g_{N} are the prescribed value and flux respectively, the vector $\boldsymbol{n}_2$ denotes the unit vector normal to Γ_2 and the coefficient $\boldsymbol{K}$ is a symmetric positive definite matrix uniformly bounded above and below. There exist constants $\lambda_{\min} > 0$ and $\lambda_{\max} > 0$ such that:

$$\text{a.e. } \boldsymbol{x} \in \Omega_2\,,\ \lambda_{\min}\boldsymbol{x}\cdot\boldsymbol{x} \le \boldsymbol{K}\boldsymbol{x}\cdot\boldsymbol{x} \le \lambda_{\max}\boldsymbol{x}\cdot\boldsymbol{x}. \tag{9}$$

The system of equations (1)-(8) is completed by interface conditions. There is no consensus in the scientific communities on the choice of these interface conditions, even for the linearized case of Stokes coupled with Darcy. We first propose to impose the the continuity of the normal component of velocity (10) and the Beaver-Joseph-Saffman [5, 28, 21] law (11) across the interface. Let $\boldsymbol{n}_{12}$ be the unit normal vector to Γ_{12} directed from Ω_1 to Ω_2 and let $\boldsymbol{\tau}_{12}$ be the unit tangent vector on Γ_{12}.

$$\begin{aligned} \boldsymbol{u}_1 \cdot \boldsymbol{n}_{12} &= -\boldsymbol{K}\nabla p_2 \cdot \boldsymbol{n}_{12}, && (10)\\ \boldsymbol{u}_1 \cdot \boldsymbol{\tau}_{12} &= -2\mu G(\boldsymbol{D}(\boldsymbol{u}_1)\boldsymbol{n}_{12}) \cdot \boldsymbol{\tau}_{12}. && (11) \end{aligned}$$

Finally, we write the balance of forces across the interface in two different ways:
(A) including inertial forces

$$((-2\mu\boldsymbol{D}(\boldsymbol{u}_1) + p_1\boldsymbol{I})\boldsymbol{n}_{12}) \cdot \boldsymbol{n}_{12} + \frac{1}{2}(\boldsymbol{u}_1 \cdot \boldsymbol{u}_1) = p_2, \tag{12}$$

(B) without inertial forces

$$((-2\mu\boldsymbol{D}(\boldsymbol{u}_1) + p_1\boldsymbol{I})\boldsymbol{n}_{12}) \cdot \boldsymbol{n}_{12} = p_2, \tag{13}$$

Condition (12) arises naturally from the momentum equation written in divergence form. Even though we are not using the divergence form of the momentum equation, we consider this condition as it is mathematically well-suited. With (12) we can prove an unconditional existence of a weak solution.

Condition (13) is the usual condition employed in the linear case of Stokes coupled with Darcy (see for instance [12, 11, 22, 27, 23, 24, 13, 18, 6]). One may argue that this condition is more physical. However, condition (13) requires additional assumptions on the data to prove existence of the solutions, and uniqueness can only be obtained locally.

The objective of the paper is to shed more light on the resulting two problems. One problem employs an interface condition that is less physical but more mathematical whereas the other problem employs an interface condition that is less mathematical but more physical. In this work we point out the differences in the mathematical analysis and we provide some numerical comparisons.

The rest of the paper is as follows. Existence and uniqueness of weak solutions are obtained in Section 2. A multinumerics approach is proposed in Section 3. Theoretical error estimates are derived in Section 4. Conclusions are given in the last section.

2 Variational Formulation

Let $H^s(\mathcal{O})$ be the usual Sobolev space of order s (see [1]) with norm $\|\cdot\|_{H^s(\mathcal{O})}$. We first lift the Dirichlet boundary condition (7). If $g_{\mathrm{D}} \in H^{1/2}_{00}(\Gamma_{2\mathrm{D}})$, there exists a function $p_{\mathrm{D}} \in H^1(\Omega_2)$ satisfying:

$$p_{\mathrm{D}} = g_{\mathrm{D}}, \text{ on } \Gamma_{2\mathrm{D}}, \tag{14}$$
$$p_{\mathrm{D}} = 0, \text{ on } \Gamma_{12}, \tag{15}$$
$$||p_{\mathrm{D}}||_{H^1(\Omega_2)} \le C_0 ||g_{\mathrm{D}}||_{H^{1/2}(\Gamma_{2\mathrm{D}})}, \tag{16}$$

where C_0 is a constant that only depends on Ω_2. We now define the standard Sobolev spaces:

$$\boldsymbol{X}_1 = \{\boldsymbol{v}_1 \in (H^1(\Omega_1))^2 : \boldsymbol{v}_1 = 0 \text{ on } \Gamma_1\},$$
$$M_1 = L^2(\Omega_1),$$
$$M_2 = \{q_2 \in H^1(\Omega_2) : q_2 = 0 \text{ on } \Gamma_{2\mathrm{D}}\}.$$

We propose the following variational formulations:

$$(W_A)\begin{cases} \text{ Find } \boldsymbol{u}_1 \in \boldsymbol{X}_1, p_1 \in M_1, p_2 = \varphi_2 + p_{\mathrm{D}}, \text{ with } \varphi_2 \in M_2, \text{ s.t.} \\ \forall \boldsymbol{v}_1 \in \boldsymbol{X}_1, \forall q_2 \in M_2,\ 2\mu\big(\boldsymbol{D}(\boldsymbol{u}_1), \boldsymbol{D}(\boldsymbol{v}_1)\big)_{\Omega_1} + \big(\boldsymbol{u}_1 \cdot \nabla \boldsymbol{u}_1, \boldsymbol{v}_1\big)_{\Omega_1} - \big(p_1, \nabla \cdot \boldsymbol{v}_1\big)_{\Omega_1} \\ +\big(\varphi_2 - \frac{1}{2}\boldsymbol{u}_1 \cdot \boldsymbol{u}_1, \boldsymbol{v}_1 \cdot \boldsymbol{n}_{12}\big)_{\Gamma_{12}} + \frac{1}{G}\big(\boldsymbol{u}_1 \cdot \boldsymbol{\tau}_{12}, \boldsymbol{v}_1 \cdot \boldsymbol{\tau}_{12}\big)_{\Gamma_{12}} - \big(\boldsymbol{u}_1 \cdot \boldsymbol{n}_{12}, q_2\big)_{\Gamma_{12}} + \big(\boldsymbol{K}\nabla\varphi_2, \nabla q_2)_{\Omega_2} \\ = \big(\boldsymbol{f}_1, \boldsymbol{v}_1\big)_{\Omega_1} + \big(f_2, q_2\big)_{\Omega_2} - \big(\boldsymbol{K}\nabla p_{\mathrm{D}}, \nabla q_2\big)_{\Omega_2} + \big(g_{\mathrm{N}}, q_2\big)_{\Gamma_{2\mathrm{N}}}, \\ \forall q_1 \in M_1,\ \ (\nabla \cdot \boldsymbol{u}_1, q_1)_{\Omega_1} = 0. \end{cases}$$

$$(W_B)\begin{cases} \text{ Find } \boldsymbol{u}_1 \in \boldsymbol{X}_1, p_1 \in M_1, p_2 = \varphi_2 + p_{\mathrm{D}}, \text{ with } \varphi_2 \in M_2, \text{ s.t.} \\ \forall \boldsymbol{v}_1 \in \boldsymbol{X}_1, \forall q_2 \in M_2,\ 2\mu\big(\boldsymbol{D}(\boldsymbol{u}_1), \boldsymbol{D}(\boldsymbol{v}_1)\big)_{\Omega_1} + \big(\boldsymbol{u}_1 \cdot \nabla \boldsymbol{u}_1, \boldsymbol{v}_1\big)_{\Omega_1} - \big(p_1, \nabla \cdot \boldsymbol{v}_1\big)_{\Omega_1} \\ +\big(\varphi_2, \boldsymbol{v}_1 \cdot \boldsymbol{n}_{12}\big)_{\Gamma_{12}} + \frac{1}{G}\big(\boldsymbol{u}_1 \cdot \boldsymbol{\tau}_{12}, \boldsymbol{v}_1 \cdot \boldsymbol{\tau}_{12}\big)_{\Gamma_{12}} - \big(\boldsymbol{u}_1 \cdot \boldsymbol{n}_{12}, q_2\big)_{\Gamma_{12}} + \big(\boldsymbol{K}\nabla\varphi_2, \nabla q_2)_{\Omega_2} \\ = \big(\boldsymbol{f}_1, \boldsymbol{v}_1\big)_{\Omega_1} + \big(f_2, q_2\big)_{\Omega_2} - \big(\boldsymbol{K}\nabla p_{\mathrm{D}}, \nabla q_2\big)_{\Omega_2} + \big(g_{\mathrm{N}}, q_2\big)_{\Gamma_{2\mathrm{N}}}, \\ \forall q_1 \in M_1,\ \ (\nabla \cdot \boldsymbol{u}_1, q_1)_{\Omega_1} = 0. \end{cases}$$

Problems (W_A) and (W_B) are very similar as they differ only by one term, namely $(\frac{1}{2}\boldsymbol{u}_1 \cdot \boldsymbol{u}_1, \boldsymbol{v}_1 \cdot \boldsymbol{n}_{12}\big)_{\Gamma_{12}}$ arising from condition (12). We have used the notation $(\cdot,\cdot)_{\mathcal{O}}$ for the L^2 inner-product on a region $\mathcal{O}$. We recall the usual Cauchy-Schwarz and Young's inequalities:

$$\forall v, w \in L^2(\mathcal{O}), \quad |(v,w)_{\mathcal{O}}| \le \|v\|_{L^2(\mathcal{O})}\|w\|_{L^2(\mathcal{O})}, \tag{17}$$
$$\forall a, b \in \mathbb{R}, \forall \delta > 0, \quad ab \le \frac{\delta}{2}a^2 + \frac{1}{2\delta}b^2. \tag{18}$$

We also recall Poincaré and Korn's inequalities and trace and Sobolev inequalities: there exist constants $\mathcal{P}_1$, C_1, C_2, C_4 and $\mathcal{P}_4$, that only depend on Ω_1, and $\mathcal{P}_2, C_3$ that only depend on Ω_2, such that for all $\boldsymbol{v} \in \boldsymbol{X}_1$,

$$\|\boldsymbol{v}\|_{L^2(\Omega_1)} \le \mathcal{P}_1 \|\nabla \boldsymbol{v}\|_{L^2(\Omega_1)}, \quad \|\boldsymbol{v}\|_{L^4(\Omega_1)} \le \mathcal{P}_4 \|\nabla \boldsymbol{v}\|_{L^2(\Omega_1)}\,, \tag{19}$$
$$\|\nabla \boldsymbol{v}\|_{L^2(\Omega_1)} \le C_1 \|\boldsymbol{D}(\boldsymbol{v})\|_{L^2(\Omega_1)}\,, \tag{20}$$
$$\|\boldsymbol{v}\|_{L^2(\Gamma_{12})} \le C_2 \|\nabla \boldsymbol{v}\|_{L^2(\Omega_1)}, \quad \|\boldsymbol{v}\|_{L^4(\Gamma_{12})} \le C_4 \|\nabla \boldsymbol{v}\|_{L^2(\Omega_1)}\,, \tag{21}$$

and for all $q \in M_2$,

$$\|q\|_{L^2(\Omega_2)} \le \mathcal{P}_2 \|\nabla q\|_{L^2(\Omega_2)}, \tag{22}$$
$$\|q\|_{L^2(\Gamma_{2\mathrm{N}})} \le C_3 \|\nabla q\|_{L^2(\Omega_2)}; \tag{23}$$

moreover, owing to (9), for all $q \in H^1(\Omega_2)$:

$$\frac{1}{\sqrt{\lambda_{\max}}}\|\boldsymbol{K}^{1/2}\nabla q\|_{L^2(\Omega_2)} \le \|\nabla q\|_{L^2(\Omega_2)} \le \frac{1}{\sqrt{\lambda_{\min}}}\|\boldsymbol{K}^{1/2}\nabla q\|_{L^2(\Omega_2)}\,. \tag{24}$$

We first show that variational formulations and corresponding model problems are equivalent.

Lemma 1. *If $(\boldsymbol{u}_1, p_1, p_2) \in \boldsymbol{X}_1 \times M_1 \times H^1(\Omega_2)$ satisfies (1)-(12), then it is also a solution to problem (W_A). If $(\boldsymbol{u}_1, p_1, p_2) \in \boldsymbol{X}_1 \times M_1 \times H^1(\Omega_2)$ satisfies (1)-(11) and (13), then it is also a solution to problem (W_B). The converse of both statements is also true.*

Proof. We first consider the problem (1)-(12) with solution $(\boldsymbol{u}_1, p_1, p_2) \in \boldsymbol{X}_1 \times M_1 \times H^1(\Omega_2)$. Multiply (1), (2) and (5) by test functions $\boldsymbol{v}_1 \in \boldsymbol{X}_1, q_1 \in M_1$ and $q_2 \in M_2$ respectively and use Green's theorem and boundary conditions:

$$\begin{aligned} 2\mu(\boldsymbol{D}(\boldsymbol{u}_1), \boldsymbol{D}(\boldsymbol{v}_1))_{\Omega_1} - (p_1, \nabla \cdot \boldsymbol{v}_1)_{\Omega_1} + (\boldsymbol{u}_1 \cdot \nabla \boldsymbol{u}_1, \boldsymbol{v}_1)_{\Omega_1} \\ +((-2\mu \boldsymbol{D}(\boldsymbol{u}_1) + p_1 \boldsymbol{I})\boldsymbol{n}_{12}, \boldsymbol{v}_1)_{\Gamma_{12}} = (\boldsymbol{f}_1, \boldsymbol{v}_1)_{\Omega_1}, \end{aligned} \tag{25}$$

$$(\nabla \cdot \boldsymbol{u}_1, q_1)_{\Omega_1} = 0, \tag{26}$$

$$(\boldsymbol{K}\nabla p_2, \nabla q_2)_{\Omega_2} + (\boldsymbol{K}\nabla p_2 \cdot \boldsymbol{n}_{12}, q_2)_{\Gamma_{12}} = (f_2, q_2)_{\Omega_2} + (g_{\mathrm{N}}, q_2)_{\Gamma_{2\mathrm{N}}}. \tag{27}$$

Rewriting $\boldsymbol{v}_1 = (\boldsymbol{v}_1 \cdot \boldsymbol{n}_{12})\boldsymbol{n}_{12} + (\boldsymbol{v}_1 \cdot \boldsymbol{\tau}_{12})\boldsymbol{\tau}_{12}$, adding (25) and (27) and using the interface conditions, we obtain:

$$\begin{aligned} 2\mu(\boldsymbol{D}(\boldsymbol{u}_1), \boldsymbol{D}(\boldsymbol{v}_1))_{\Omega_1} - (p_1, \nabla \cdot \boldsymbol{v}_1)_{\Omega_1} + (\boldsymbol{u}_1 \cdot \nabla \boldsymbol{u}_1, \boldsymbol{v}_1)_{\Omega_1} + (\boldsymbol{K}\nabla p_2, \nabla q_2)_{\Omega_2} + \big(p_2 - \frac{1}{2}\boldsymbol{u}_1 \cdot \boldsymbol{u}_1, \boldsymbol{v}_1 \cdot \boldsymbol{n}_{12}\big)_{\Gamma_{12}} \\ +\frac{1}{G}\big(\boldsymbol{u}_1 \cdot \boldsymbol{\tau}_{12}, \boldsymbol{v}_1 \cdot \boldsymbol{\tau}_{12}\big)_{\Gamma_{12}} - \big(\boldsymbol{u}_1 \cdot \boldsymbol{n}_{12}, q_2\big)_{\Gamma_{12}} = (\boldsymbol{f}_1, \boldsymbol{v})_{\Omega_1} + (f_2, q_2)_{\Omega_2} + (g_{\mathrm{N}}, q_2)_{\Gamma_{2\mathrm{N}}}, \\ (\nabla \cdot \boldsymbol{u}_1, q_1)_{\Omega_1} = 0. \end{aligned}$$

We now define $\varphi_2 = p_2 - p_{\mathrm{D}}$ and remark that the trace $p_2 = \varphi_2$ on Γ_{12} due to (15). We obtain the resulting equations:

$$\begin{aligned} 2\mu(\boldsymbol{D}(\boldsymbol{u}_1), \boldsymbol{D}(\boldsymbol{v}_1))_{\Omega_1} - (p_1, \nabla \cdot \boldsymbol{v}_1)_{\Omega_1} + (\boldsymbol{u}_1 \cdot \nabla \boldsymbol{u}_1, \boldsymbol{v}_1)_{\Omega_1} + (\boldsymbol{K}\nabla \varphi_2, \nabla q_2)_{\Omega_2} + \big(\varphi_2 - \frac{1}{2}\boldsymbol{u}_1 \cdot \boldsymbol{u}_1, \boldsymbol{v}_1 \cdot \boldsymbol{n}_{12}\big)_{\Gamma_{12}} \\ +\frac{1}{G}\big(\boldsymbol{u}_1 \cdot \boldsymbol{\tau}_{12}, \boldsymbol{v}_1 \cdot \boldsymbol{\tau}_{12}\big)_{\Gamma_{12}} - \big(\boldsymbol{u}_1 \cdot \boldsymbol{n}_{12}, q_2\big)_{\Gamma_{12}} = (\boldsymbol{f}_1, \boldsymbol{v}_1)_{\Omega_1} + (f_2, q_2)_{\Omega_2} + (g_{\mathrm{N}}, q_2)_{\Gamma_{2\mathrm{N}}} - (\boldsymbol{K}\nabla p_{\mathrm{D}}, \nabla q_2)_{\Omega_2}, \\ (\nabla \cdot \boldsymbol{u}_1, q_1)_{\Omega_1} = 0, \end{aligned}$$

which correspond to problem (W_A). Conversely, assume that $(\boldsymbol{u}_1, p_1, p_2)$ is a solution to (W_A). By choosing appropriate test functions, we recover the equations (1), (2) and (5) in a distributional sense. First, take $\boldsymbol{v}_1 \in \mathcal{D}(\Omega_1)$, $q_1 = q_2 = 0$. We recall that for any domain $\mathcal{O}$, the space $\mathcal{D}(\mathcal{O})$ is the space of $\mathcal{C}^\infty$ functions with compact support in $\mathcal{O}$ (see [1]). We obtain in the sense of distributions:

$$-\nabla \cdot (2\mu D(\boldsymbol{u}_1) - p_1 \boldsymbol{I}) + \boldsymbol{u}_1 \cdot \nabla \boldsymbol{u}_1 = \boldsymbol{f}_1. \tag{28}$$

Second, take $q_1 \in \mathcal{D}(\Omega_1)$, $\boldsymbol{v}_1 = \boldsymbol{0}$, $q_2 = 0$:

$$\nabla \cdot \boldsymbol{u}_1 = 0. \tag{29}$$

Third, take $q_2 \in \mathcal{D}(\Omega_2)$, $\boldsymbol{v}_1 = \boldsymbol{0}$, $q_1 = 0$:

$$-\nabla \cdot \boldsymbol{K}\nabla(\varphi_2 + p_{\mathrm{D}}) = f_2. \tag{30}$$

Next, multiply (28), (30) by functions $\boldsymbol{v}_1 \in \boldsymbol{X}_1$ and $q_2 \in M_2$ respectively, use Green's theorem, add the two equations and compare with (W_A):

$$\begin{aligned} (\varphi_2 - \frac{1}{2}(\boldsymbol{u}_1 \cdot \boldsymbol{u}_1), \boldsymbol{v}_1 \cdot \boldsymbol{n}_{12})_{\Gamma_{12}} - (\boldsymbol{u}_1 \cdot \boldsymbol{n}_{12}, q_2)_{\Gamma_{12}} + \frac{1}{G}(\boldsymbol{u}_1 \cdot \boldsymbol{\tau}_{12}, \boldsymbol{v}_1 \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} - (g_{\mathrm{N}}, q_2)_{\Gamma_{2\mathrm{N}}} \\ = ((-2\mu \boldsymbol{D}(\boldsymbol{u}_1) + p_1 \boldsymbol{I})\boldsymbol{n}_{12}, \boldsymbol{v}_1)_{\Gamma_{12}} + (\boldsymbol{K}\nabla p_2 \cdot \boldsymbol{n}_{12}, q_2)_{\Gamma_{12}} - (\boldsymbol{K}\nabla p_2 \cdot \boldsymbol{n}_2, q_2)_{\Gamma_{2\mathrm{N}}}. \end{aligned} \tag{31}$$

By choosing $\boldsymbol{v}_1 = \boldsymbol{0}$ and either $q_2|_{\Gamma_{12}} = 0$ or $q_2|_{\Gamma_{2\mathrm{N}}} = 0$, we recover the Neumann boundary condition (8) and the interface condition (10). Next, by choosing $q_2 = 0$ and $\boldsymbol{v}_1 = v_1 \boldsymbol{n}_{12}$ where v_1 is a smooth function defined on each curvilinear segment of Γ_{12} and vanishing in a neighborhood of $\partial\Omega_1 \setminus \Gamma_{12}$, we recover the interface condition (12) by noting that $p_2 = \varphi_2$ on Γ_{12} due to (15). Finally, choosing $q_2 = 0$ and $\boldsymbol{v}_1 = v_1 \boldsymbol{\tau}_{12}$ where v_1 is a smooth function defined on each curvilinear segment of Γ_{12} and vanishing in a neighborhood of $\partial\Omega_1 \setminus \Gamma_{12}$, we recover the interface condition (11). The equivalence of problem (W_B) and (1)-(11) with (13) is obtained in a very similar fashion. □

We now prove existence and uniqueness of the weak solutions of (W_A) and (W_B). For this, we restrict the test functions $\boldsymbol{v}_1$ to the subspace of divergence free functions:

$$\boldsymbol{V}_1 = \{\boldsymbol{v}_1 \in \boldsymbol{X}_1, \quad \nabla \cdot \boldsymbol{v}_1 = 0\}.$$

The variational formulations then become:

$$(\tilde{W}_A) \begin{cases} \text{Find } \boldsymbol{u}_1 \in \boldsymbol{V}_1, p_2 = \varphi_2 + p_\mathrm{D}, \text{ with } \varphi_2 \in M_2, \text{ s.t.} \\ \forall \boldsymbol{v}_1 \in \boldsymbol{V}_1, \forall q_2 \in M_2, \;\; 2\mu\big(\boldsymbol{D}(\boldsymbol{u}_1), \boldsymbol{D}(\boldsymbol{v}_1)\big)_{\Omega_1} + \big(\boldsymbol{u}_1 \cdot \nabla \boldsymbol{u}_1, \boldsymbol{v}_1\big)_{\Omega_1} + \big(\varphi_2 - \frac{1}{2}\boldsymbol{u}_1 \cdot \boldsymbol{u}_1, \boldsymbol{v}_1 \cdot \boldsymbol{n}_{12}\big)_{\Gamma_{12}} \\ +\frac{1}{G}\big(\boldsymbol{u}_1 \cdot \boldsymbol{\tau}_{12}, \boldsymbol{v}_1 \cdot \boldsymbol{\tau}_{12}\big)_{\Gamma_{12}} - \big(\boldsymbol{u}_1 \cdot \boldsymbol{n}_{12}, q_2\big)_{\Gamma_{12}} + \big(\boldsymbol{K}\nabla\varphi_2, \nabla q_2\big)_{\Omega_2} \\ = \big(\boldsymbol{f}_1, \boldsymbol{v}_1\big)_{\Omega_1} + \big(f_2, q_2\big)_{\Omega_2} - \big(\boldsymbol{K}\nabla p_\mathrm{D}, \nabla q_2\big)_{\Omega_2} + \big(g_\mathrm{N}, q_2\big)_{\Gamma_{2\mathrm{N}}}. \end{cases}$$

$$(\tilde{W}_B) \begin{cases} \text{Find } \boldsymbol{u}_1 \in \boldsymbol{V}_1, p_2 = \varphi_2 + p_\mathrm{D}, \text{ with } \varphi_2 \in M_2, \text{ s.t.} \\ \forall \boldsymbol{v}_1 \in \boldsymbol{V}_1, \forall q_2 \in M_2, \;\; 2\mu\big(\boldsymbol{D}(\boldsymbol{u}_1), \boldsymbol{D}(\boldsymbol{v}_1)\big)_{\Omega_1} + \big(\boldsymbol{u}_1 \cdot \nabla \boldsymbol{u}_1, \boldsymbol{v}_1\big)_{\Omega_1} + \big(\varphi_2, \boldsymbol{v}_1 \cdot \boldsymbol{n}_{12}\big)_{\Gamma_{12}} \\ +\frac{1}{G}\big(\boldsymbol{u}_1 \cdot \boldsymbol{\tau}_{12}, \boldsymbol{v}_1 \cdot \boldsymbol{\tau}_{12}\big)_{\Gamma_{12}} - \big(\boldsymbol{u}_1 \cdot \boldsymbol{n}_{12}, q_2\big)_{\Gamma_{12}} + \big(\boldsymbol{K}\nabla\varphi_2, \nabla q_2\big)_{\Omega_2} \\ = \big(\boldsymbol{f}_1, \boldsymbol{v}_1\big)_{\Omega_1} + \big(f_2, q_2\big)_{\Omega_2} - \big(\boldsymbol{K}\nabla p_\mathrm{D}, \nabla q_2\big)_{\Omega_2} + \big(g_\mathrm{N}, q_2\big)_{\Gamma_{2\mathrm{N}}}. \end{cases}$$

For $\alpha = A, B$, problems (W_α) and $(\tilde{W}_\alpha)$ are equivalent in the sense that if $(\boldsymbol{u}_1, p_1, p_2)$ is a solution to (W_α) then clearly $(\boldsymbol{u}_1, p_2)$ is also a solution to $(\tilde{W}_\alpha)$. Conversely, if $(\boldsymbol{u}_1, p_2)$ is a solution to $(\tilde{W}_\alpha)$, there is a unique $p_1 \in M_1$ such that $(\boldsymbol{u}_1, p_1, p_2)$ is a solution to (W_α). This result is a consequence of the following inf-sup condition proved in [16].

$$\inf_{q_1 \in M_1} \sup_{(\boldsymbol{v}_1, q_2) \in \boldsymbol{X}_1 \times M_2} \frac{|(\nabla \cdot \boldsymbol{v}_1, q_1)_{\Omega_1}|}{(\|\nabla \boldsymbol{v}_1\|^2_{L^2(\Omega_1)} + ||\nabla q_2||^2_{L^2(\Omega_2)})^{1/2} \|q_1\|_{L^2(\Omega_1)}} \geq \beta > 0.$$

Therefore, we now focus on the existence and uniqueness of solution to $(\tilde{W}_A)$ and to $(\tilde{W}_B)$. The proofs are similar to the ones obtained for the homogeneous problem analyzed in [16]. We give them for completeness.

2.1 Existence of solution to problem $(\tilde{W}_A)$ and problem $(\tilde{W}_B)$

We use the technique of the Galerkin method. Since the spaces $\boldsymbol{V}_1$ and M_2 are separable, let $\{(\boldsymbol{w}_m, t_m)\}_{m \geq 1}$ be a sequence of smooth functions that form a basis of $Y = \boldsymbol{V}_1 \times M_2$. Consider the finite dimensional space $Y_m = \mathrm{span}\{(\boldsymbol{w}_i, t_i) : 1 \leq i \leq m\}$ equipped with the inner-product:

$$((\boldsymbol{v}, q), (\boldsymbol{w}, t))_Y = 2\mu(\boldsymbol{D}(\boldsymbol{v}), \boldsymbol{D}(\boldsymbol{w}))_{\Omega_1} + (\boldsymbol{K}\nabla q, \nabla t)_{\Omega_2}.$$

We restrict problem $(\tilde{W}_A)$ to Y_m and obtain a finite dimensional problem:

$$(\tilde{W}_{A,m}) \begin{cases} \text{Find } (\boldsymbol{u}_m, \varphi_m) \in Y_m \text{ s.t.} \\ \forall 1 \leq i \leq m, \;\; 2\mu\big(\boldsymbol{D}(\boldsymbol{u}_m), \boldsymbol{D}(\boldsymbol{w}_i)\big)_{\Omega_1} + \big(\boldsymbol{u}_m \cdot \nabla \boldsymbol{u}_m, \boldsymbol{w}_i\big)_{\Omega_1} + \big(\varphi_m - \frac{1}{2}\boldsymbol{u}_m \cdot \boldsymbol{u}_m, \boldsymbol{w}_i \cdot \boldsymbol{n}_{12}\big)_{\Gamma_{12}} \\ +\frac{1}{G}\big(\boldsymbol{u}_m \cdot \boldsymbol{\tau}_{12}, \boldsymbol{w}_i \cdot \boldsymbol{\tau}_{12}\big)_{\Gamma_{12}} - \big(\boldsymbol{u}_m \cdot \boldsymbol{n}_{12}, t_i\big)_{\Gamma_{12}} + \big(\boldsymbol{K}\nabla\varphi_m, \nabla t_i\big)_{\Omega_2} \\ = \big(\boldsymbol{f}_1, \boldsymbol{w}_i\big)_{\Omega_1} + \big(f_2, t_i\big)_{\Omega_2} - \big(\boldsymbol{K}\nabla p_\mathrm{D}, \nabla t_i\big)_{\Omega_2} + \big(g_\mathrm{N}, t_i\big)_{\Gamma_{2\mathrm{N}}}. \end{cases}$$

We then define a continuous mapping $\Psi_{A,m} : Y_m \to Y_m$:

$$\begin{aligned}\big(\Psi_{A,m}(\boldsymbol{v},q),(\boldsymbol{w},t)\big)_Y \quad &= \quad 2\mu\big(\boldsymbol{D}(\boldsymbol{v}),\boldsymbol{D}(\boldsymbol{w})\big)_{\Omega_1} + \big(\boldsymbol{v}\cdot\nabla\boldsymbol{v},\boldsymbol{w}\big)_{\Omega_1} + \big(q - \frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v},\boldsymbol{w}\cdot\boldsymbol{n}_{12}\big)_{\Gamma_{12}} + \frac{1}{G}\big(\boldsymbol{v}\cdot\boldsymbol{\tau}_{12},\boldsymbol{w}\cdot\boldsymbol{\tau}_{12}\big)_{\Gamma_{12}}\\ &\quad -\big(\boldsymbol{v}\cdot\boldsymbol{n}_{12},t\big)_{\Gamma_{12}} + \big(\boldsymbol{K}\nabla q,\nabla t\big)_{\Omega_2} - \big(\boldsymbol{f}_1,\boldsymbol{w}\big)_{\Omega_1} - \big(f_2,t\big)_{\Omega_2} + \big(\boldsymbol{K}\nabla p_{\mathrm{D}},\nabla t\big)_{\Omega_2} - \big(g_{\mathrm{N}},t\big)_{\Gamma_{2\mathrm{N}}}.\end{aligned}$$

Clearly a zero of $\Psi_{A,m}$ is a solution to problem $(\tilde{W}_{A,m})$. We will apply a corollary of Brouwer's fixed point theorem to conclude that there is at least one zero of $\Psi_{A,m}$ in a certain ball centered at the origin. For completeness, the result is recalled below [15].

Lemma 2. *Let H be a finite dimensional Hilbert space with inner-product $(\cdot,\cdot)_H$ and norm $\|\cdot\|_H$. Let $\mathcal{F}$ be a continuous mapping from H into H. Assume there is a constant $\mathcal{R}$ such that*

$$\forall v \in H \text{ with } \|v\|_H = \mathcal{R}, \quad (\mathcal{F}(v),v)_H \geq 0.$$

Then, there exists an element $v_0 \in H$ such that

$$\mathcal{F}(v_0) = 0, \quad \|v_0\|_H \leq \mathcal{R}.$$

Therefore, we evaluate

$$\begin{aligned}\big(\Psi_{A,m}(\boldsymbol{v},q),(\boldsymbol{v},q)\big)_Y \quad &= \quad 2\mu\big(\boldsymbol{D}(\boldsymbol{v}),\boldsymbol{D}(\boldsymbol{v})\big)_{\Omega_1} + \big(\boldsymbol{v}\cdot\nabla\boldsymbol{v},\boldsymbol{v}\big)_{\Omega_1} + \big(q - \frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v},\boldsymbol{v}\cdot\boldsymbol{n}_{12}\big)_{\Gamma_{12}} + \frac{1}{G}\big(\boldsymbol{v}\cdot\boldsymbol{\tau}_{12},\boldsymbol{v}\cdot\boldsymbol{\tau}_{12}\big)_{\Gamma_{12}}\\ &\quad -\big(\boldsymbol{v}\cdot\boldsymbol{n}_{12},q\big)_{\Gamma_{12}} + \big(\boldsymbol{K}\nabla q,\nabla q\big)_{\Omega_2} - \big(\boldsymbol{f}_1,\boldsymbol{v}\big)_{\Omega_1} - \big(f_2,q\big)_{\Omega_2} + \big(\boldsymbol{K}\nabla p_{\mathrm{D}},\nabla q\big)_{\Omega_2} - \big(g_{\mathrm{N}},q\big)_{\Gamma_{2\mathrm{N}}}.\end{aligned}$$

We remark that for $\boldsymbol{v} \in \boldsymbol{V}_1$

$$\big(\boldsymbol{v}\cdot\nabla\boldsymbol{v},\boldsymbol{v}\big)_{\Omega_1} = -\frac{1}{2}\big(\nabla\cdot\boldsymbol{v},\boldsymbol{v}\cdot\boldsymbol{v}\big)_{\Omega_1} + \frac{1}{2}\big(\boldsymbol{v}\cdot\boldsymbol{n}_1,\boldsymbol{v}\cdot\boldsymbol{v}\big)_{\partial\Omega_1} = \frac{1}{2}\big(\boldsymbol{v}\cdot\boldsymbol{n}_1,\boldsymbol{v}\cdot\boldsymbol{v}\big)_{\partial\Omega_1} \tag{32}$$

Therefore,

$$\big(\boldsymbol{v}\cdot\nabla\boldsymbol{v},\boldsymbol{v}\big)_{\Omega_1} + \big(q - \frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v},\boldsymbol{v}\cdot\boldsymbol{n}_{12}\big)_{\Gamma_{12}} - \big(\boldsymbol{v}\cdot\boldsymbol{n}_{12},q\big)_{\Gamma_{12}} = 0,$$

because $\boldsymbol{v} = 0$ on Γ_1. We are left with

$$\begin{aligned}\big(\Psi_{A,m}(\boldsymbol{v},q),(\boldsymbol{v},q)\big)_Y \quad &= \quad 2\mu\|\boldsymbol{D}(\boldsymbol{v})\|^2_{L^2(\Omega_1)} + \frac{1}{G}\|\boldsymbol{v}\cdot\boldsymbol{\tau}_{12}\|^2_{L^2(\Gamma_{12})} + \|\boldsymbol{K}^{1/2}\nabla q\|^2_{L^2(\Omega_2)}\\ &\quad -\big(\boldsymbol{f}_1,\boldsymbol{v}\big)_{\Omega_1} - \big(f_2,q\big)_{\Omega_2} + \big(\boldsymbol{K}\nabla p_{\mathrm{D}},\nabla q\big)_{\Omega_2} - \big(g_{\mathrm{N}},q\big)_{\Gamma_{2\mathrm{N}}}.\end{aligned} \tag{33}$$

We now bound the terms in the second line of (33). Using (17), (19), (20) and (18), we obtain

$$\begin{aligned}|\big(\boldsymbol{f}_1,\boldsymbol{v}\big)_{\Omega_1}| &\leq ||\boldsymbol{f}_1||_{L^2(\Omega_1)}||\boldsymbol{v}||_{L^2(\Omega_1)} \leq \mathcal{P}_1 C_1||\boldsymbol{D}(\boldsymbol{v})||_{L^2(\Omega_1)}||\boldsymbol{f}_1||_{L^2(\Omega_1)}\\ &\leq \frac{\mu}{2}||\boldsymbol{D}(\boldsymbol{v})||^2_{L^2(\Omega_1)} + \frac{\mathcal{P}_1^2 C_1^2}{2\mu}||\boldsymbol{f}_1||^2_{L^2(\Omega_1)}.\end{aligned} \tag{34}$$

Similarly, using (17), (22), (24) and (18), we have

$$|\big(f_2,q\big)_{\Omega_2}| \leq \frac{1}{4}||\boldsymbol{K}^{1/2}\nabla q||^2_{L^2(\Omega_2)} + \frac{1}{\lambda_{\min}}\mathcal{P}_2^2||f_2||^2_{L^2(\Omega_2)}. \tag{35}$$

Using the bounds (16), (17) and (24), we have

$$|\big(\boldsymbol{K}\nabla p_{\mathrm{D}},\nabla q\big)_{\Omega_2}| \leq \frac{1}{4}||\boldsymbol{K}^{1/2}\nabla q||^2_{L^2(\Omega_2)} + C_0^2\lambda_{\max}||g_{\mathrm{D}}||^2_{H^{1/2}(\Gamma_{2\mathrm{D}})}. \tag{36}$$

Finally, using (17), (23), (24) and (18), we obtain

$$|\big(g_{\rm N},q\big)_{\Gamma_{2\rm N}}| \le \frac{1}{4}||\boldsymbol{K}^{1/2}\nabla q||^2_{L^2(\Omega_2)} + \frac{C_3^2}{\lambda_{\min}}||g_{\rm N}||^2_{L^2(\Gamma_{2\rm N})}. \tag{37}$$

Therefore

$$\big(\Psi_{A,m}(\boldsymbol{v},q),(\boldsymbol{v},q)\big)_Y \ge \frac{1}{4}\Big(2\mu||\boldsymbol{D}(\boldsymbol{v})||^2_{L^2(\Omega_1)} + ||\boldsymbol{K}^{1/2}\nabla q||^2_{L^2(\Omega_2)}\Big) - \Big(\frac{\mathcal{P}_1^2C_1^2}{2\mu}||\boldsymbol{f}_1||^2_{L^2(\Omega_1)} + \frac{\mathcal{P}_2^2}{\lambda_{\min}}||f_2||^2_{L^2(\Omega_2)}$$
$$+C_0^2\lambda_{\max}||g_{\rm D}||^2_{H^{1/2}(\Gamma_{2\rm D})} + \frac{C_3^2}{\lambda_{\min}}||g_{\rm N}||^2_{L^2(\Gamma_{2\rm N})}\Big),$$

so $\big(\Psi_{A,m}(\boldsymbol{v},q),(\boldsymbol{v},q)\big)_Y \ge 0$ provided $\|(\boldsymbol{v},q)\|_Y = ((\boldsymbol{v},q),(\boldsymbol{v},q))_Y^{1/2} = \mathcal{R}_0$ with

$$\mathcal{R}_0 = 2\Big(\frac{\mathcal{P}_1^2C_1^2}{2\mu}||\boldsymbol{f}_1||^2_{L^2(\Omega_1)} + \frac{\mathcal{P}_2^2}{\lambda_{\min}}||f_2||^2_{L^2(\Omega_2)} + C_0^2\lambda_{\max}||g_{\rm D}||^2_{H^{1/2}(\Gamma_{2\rm D})} + \frac{C_3^2}{\lambda_{\min}}||g_{\rm N}||^2_{L^2(\Gamma_{2\rm N})}\Big)^{1/2}. \tag{38}$$

Therefore, for any m, there is a solution $(\boldsymbol{u}_m,\varphi_m)$ of problem $(\tilde{W}_{A,m})$ satisfying:

$$\|(\boldsymbol{u}_m,\varphi_m)\|_Y \le \mathcal{R}_0.$$

We have thus constructed a bounded sequence in the Hilbert space $\boldsymbol{V}_1 \times M_2$. Therefore, there exists a subsequence, still denoted by $\{(\boldsymbol{u}_m,\varphi_m)\}_m$, that converges weakly to an element $(\boldsymbol{u},\varphi) \in \boldsymbol{V}_1 \times M_2$. Using a standard argument and Sobolev imbeddings, we can pass to the limit in the equation of problem $(\tilde{W}_{A,m})$ as m tends to infinity. Denoting $p = \varphi + p_{\rm D}$, we then obtain that $(\boldsymbol{u},p)$ is a solution to problem $(\tilde{W}_A)$. Using the same argument as above, we can show that any solution $(\boldsymbol{u},\varphi)$ to problem $(\tilde{W}_A)$ is bounded:

$$2\mu\|\boldsymbol{D}(\boldsymbol{u})\|^2_{L^2(\Omega_1)} + \|\boldsymbol{K}^{1/2}\nabla\varphi\|^2_{L^2(\Omega_2)} \le \mathcal{R}_0^2. \tag{39}$$

This yields the bound:

$$2\mu\|\boldsymbol{D}(\boldsymbol{u})\|^2_{L^2(\Omega_1)} + \|\boldsymbol{K}^{1/2}\nabla p\|^2_{L^2(\Omega_2)} \le \mathcal{R}_1^2, \tag{40}$$

where

$$\mathcal{R}_1^2 = \mathcal{R}_0^2 + 2\|\boldsymbol{K}^{1/2}\nabla p_{\rm D}\|^2_{L^2(\Omega_2)}. \tag{41}$$

To summarize, we have proved the following lemma.

Lemma 3. *There is a solution to problem $(\tilde{W}_A)$ satisfying (39).*

Next, we show existence of a solution to $(\tilde{W}_B)$ under a small data condition.

Lemma 4. *Let $\mathcal{R}_0$ defined by (38). Assume that*

$$\mathcal{R}_0^2 < \frac{2\mu^3}{C_1^6\mathcal{P}_4^4}. \tag{42}$$

Then, there exists a solution to problem $(\tilde{W}_B)$ satisfying (39).

Proof. As in the proof above, we define a sequence of finite-dimensional problems $(W_{B,m})$ whose solutions converge to a solution to (W_B).

$$(\tilde{W}_{B,m})\begin{cases} \text{Find } (\boldsymbol{u}_m,\varphi_m) \in Y_m \text{ s.t.} \\ \forall 1 \le i \le m,\ \ 2\mu\big(\boldsymbol{D}(\boldsymbol{u}_m),\boldsymbol{D}(\boldsymbol{w}_i)\big)_{\Omega_1} + \big(\boldsymbol{u}_m\cdot\nabla\boldsymbol{u}_m,\boldsymbol{w}_i\big)_{\Omega_1} + \big(\varphi_m,\boldsymbol{w}_i\cdot\boldsymbol{n}_{12}\big)_{\Gamma_{12}} \\ +\dfrac{1}{G}\big(\boldsymbol{u}_m\cdot\boldsymbol{\tau}_{12},\boldsymbol{w}_i\cdot\boldsymbol{\tau}_{12}\big)_{\Gamma_{12}} - \big(\boldsymbol{u}_m\cdot\boldsymbol{n}_{12},t_i\big)_{\Gamma_{12}} + \big(\boldsymbol{K}\nabla\varphi_m,\nabla t_i\big)_{\Omega_2} \\ = \big(\boldsymbol{f}_1,\boldsymbol{w}_i\big)_{\Omega_1} + \big(f_2,t_i\big)_{\Omega_2} - \big(\boldsymbol{K}\nabla p_{\rm D},\nabla t_i\big)_{\Omega_2} + \big(g_{\rm N},t_i\big)_{\Gamma_{2\rm N}}. \end{cases}$$

This yields the following continuous mapping $\Psi_{B,m} : Y_m \to Y_m$:

$$\begin{aligned}\big(\Psi_{B,m}(\boldsymbol{v},q),(\boldsymbol{w},t)\big)_Y &= 2\mu\big(\boldsymbol{D}(\boldsymbol{v}),\boldsymbol{D}(\boldsymbol{w})\big)_{\Omega_1} + \big(\boldsymbol{v}\cdot\nabla\boldsymbol{v},\boldsymbol{w}\big)_{\Omega_1} + \big(q,\boldsymbol{w}\cdot\boldsymbol{n}_{12}\big)_{\Gamma_{12}} + \frac{1}{G}\big(\boldsymbol{v}\cdot\boldsymbol{\tau}_{12},\boldsymbol{w}\cdot\boldsymbol{\tau}_{12}\big)_{\Gamma_{12}}\\ &\quad -\big(\boldsymbol{v}\cdot\boldsymbol{n}_{12},t\big)_{\Gamma_{12}} + \big(\boldsymbol{K}\nabla q,\nabla t\big)_{\Omega_2} - \big(\boldsymbol{f}_1,\boldsymbol{w}\big)_{\Omega_1} - \big(f_2,t\big)_{\Omega_2} + \big(\boldsymbol{K}\nabla p_{\mathrm{D}},\nabla t\big)_{\Omega_2} - \big(g_{\mathrm{N}},t\big)_{\Gamma_{2\mathrm{N}}}.\end{aligned}$$

As above, we evaluate $\big(\Psi_{B,m}(\boldsymbol{v},q),(\boldsymbol{v},q)\big)_Y$ and use (34), (35), (36), (37) as well as the bound:

$$(\boldsymbol{v}\cdot\nabla\boldsymbol{v},\boldsymbol{v})_{\Omega_1} \le C_1^3\mathcal{P}_4^2\|\boldsymbol{D}(\boldsymbol{v})\|^3_{L^2(\Omega_1)}.$$

A simple calculation shows that if

$$2\mu\|\boldsymbol{D}(\boldsymbol{v})\|^2_{L^2(\Omega_1)} \le \frac{2\mu^3}{C_1^6\mathcal{P}_4^4}, \tag{43}$$

then, using (38) we have

$$\big(\Psi_{B,m}(\boldsymbol{v},q),(\boldsymbol{v},q)\big)_Y \ge \frac{1}{4}\big(\|(\boldsymbol{v},q)\|_Y^2 - \mathcal{R}_0^2\big).$$

Therefore, if the following condition holds

$$\mathcal{R}_0^2 < \frac{2\mu^3}{C_1^6\mathcal{P}_4^4}, \tag{44}$$

there is a ball of radius $\mathcal{R}_0$ on which $\big(\Psi_{B,m}(\boldsymbol{v},q),(\boldsymbol{v},q)\big)_Y \ge 0$. We thus obtain a solution $(\boldsymbol{u}_m,\varphi_m)$ of problem $(W_{B,m})$, that lies inside this ball and we obtain a solution to problem (W_B) by passing to the limit. This solution also satisfies the bound (39). □

2.2 Uniqueness of solution to problem $(\tilde{W}_A)$ and problem $(\tilde{W}_B)$

Lemma 5. *Assume that the data satisfies:*

$$\begin{aligned}\frac{16\mu^3}{C_1^6(2\mathcal{P}_4^2+3C_4^2C_2)^2} \quad > \quad & \frac{2\mathcal{P}_1^2C_1^2}{\mu}||\mathbf{f}_1||^2_{L^2(\Omega_1)} + \frac{4\mathcal{P}_2^2}{\lambda_{\min}}||f_2||^2_{L^2(\Omega_2)}\\ & + 4C_0^2\lambda_{\max}||g_{\mathrm{D}}||^2_{H^{1/2}(\Gamma_{2\mathrm{D}})} + \frac{4C_3^2}{\lambda_{\min}}||g_{\mathrm{N}}||^2_{L^2(\Gamma_{2\mathrm{N}})}.\end{aligned}$$

Then problem $(\tilde{W}_A)$ has a unique weak solution.

Proof. Assume that $(\boldsymbol{u}_1^1,p_2^1)$ and $(\boldsymbol{u}_1^2,p_2^2)$ are two solutions of problem $(\tilde{W}_A)$. Their difference, say $(\boldsymbol{w}_1,z_2)$, belongs to the space $\boldsymbol{V}_1\times M_2$ and satisfies:

$$\begin{aligned}&\forall(\boldsymbol{v}_1,q_2)\in\boldsymbol{V}_1\times M_2,\quad 2\mu(\boldsymbol{D}(\boldsymbol{w}_1),\boldsymbol{D}(\boldsymbol{v}_1))_{\Omega_1} + (\boldsymbol{K}\nabla z_2,\nabla q_2)_{\Omega_2} + (\boldsymbol{w}_1\cdot\nabla\boldsymbol{u}_1^1,\boldsymbol{v}_1)_{\Omega_1} + (\boldsymbol{u}_1^2\cdot\nabla\boldsymbol{w}_1,\boldsymbol{v}_1)_{\Omega_1}\\ &+\frac{1}{G}(\boldsymbol{w}_1\cdot\boldsymbol{\tau}_{12},\boldsymbol{v}_1\cdot\boldsymbol{\tau}_{12})_{\Gamma_{12}} + (z_2 - \frac{1}{2}(\boldsymbol{w}_1\cdot\boldsymbol{u}_1^1),\boldsymbol{v}_1\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} - (\boldsymbol{w}_1\cdot\boldsymbol{n}_{12},q_2)_{\Gamma_{12}} - \frac{1}{2}(\boldsymbol{u}_1^2\cdot\boldsymbol{w}_1,\boldsymbol{v}_1\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} = 0\,.\end{aligned}$$

By choosing $(\boldsymbol{v}_1,q_2)=(\boldsymbol{w}_1,z_2)\in\boldsymbol{V}_1\times M_2$ and applying Green's formula and the boundary condition on the functions of $\boldsymbol{X}_1$, this equation becomes

$$\begin{aligned}&2\mu\|\boldsymbol{D}(\boldsymbol{w}_1)\|^2_{L^2(\Omega_1)} + \|\boldsymbol{K}^{1/2}\nabla z_2\|^2_{L^2(\Omega_2)} + \frac{1}{G}\|\boldsymbol{w}_1\cdot\boldsymbol{\tau}_{12}\|^2_{L^2(\Gamma_{12})}\\ &\qquad + (\boldsymbol{w}_1\cdot\nabla\boldsymbol{u}_1^1,\boldsymbol{w}_1)_{\Omega_1} + \frac{1}{2}\Big((\boldsymbol{w}_1\cdot\boldsymbol{w}_1,\boldsymbol{u}_1^2\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} - (\boldsymbol{w}_1\cdot(\boldsymbol{u}_1^1+\boldsymbol{u}_1^2),\boldsymbol{w}_1\cdot\boldsymbol{n}_{12})_{\Gamma_{12}}\Big) = 0\,.\end{aligned} \tag{45}$$

Applying (19) and (20), the first non-linear term in the second line of (45) is bounded above by

$$\|\boldsymbol{w}_1\|^2_{L^4(\Omega_1)}\|\nabla\boldsymbol{u}_1^1\|_{L^2(\Omega_1)} \le C_1^3\mathcal{P}_4^2\frac{1}{\sqrt{\mu}}\|\boldsymbol{D}(\boldsymbol{w}_1)\|^2_{L^2(\Omega_1)}\left(\sqrt{\mu}\|\boldsymbol{D}(\boldsymbol{u}_1^1)\|_{L^2(\Omega_1)}\right).$$

Similarly, applying formulas (19)–(21), the second term in the second line of (45) is bounded above by

$$\frac{1}{2}\|\boldsymbol{w}_1\|^2_{L^4(\Gamma_{12})}\left(\|\boldsymbol{u}_1^1\|_{L^2(\Gamma_{12})}+2\|\boldsymbol{u}_1^2\|_{L^2(\Gamma_{12})}\right)$$
$$\leq \frac{1}{2}C_4^2 C_2 C_1^3 \frac{1}{\sqrt{\mu}}\|\boldsymbol{D}(\boldsymbol{w}_1)\|^2_{L^2(\Omega_1)}\left(\sqrt{\mu}\|\boldsymbol{D}(\boldsymbol{u}_1^1)\|_{L^2(\Omega_1)}+2\sqrt{\mu}\|\boldsymbol{D}(\boldsymbol{u}_1^2)\|_{L^2(\Omega_1)}\right).$$

Hence, using the a priori estimate (39), the second line in (45) is bounded above by

$$\frac{C_1^3}{\sqrt{2\mu}}\left(\mathcal{P}_4^2+\frac{3}{2}C_4^2C_2\right)\mathcal{R}_0\|\boldsymbol{D}(\boldsymbol{w}_1)\|^2_{L^2(\Omega_1)}.$$

Thus if

$$(2\mu)^{3/2} > C_1^3\left(\mathcal{P}_4^2+\frac{3}{2}C_4^2C_2\right)\mathcal{R}_0,$$

then $(\boldsymbol{w}_1, z_2) = (\boldsymbol{0}, 0)$. □

We cannot show that any solution to problem (W_B) is bounded. Therefore, we can only prove uniqueness of the solution inside a certain ball.

Lemma 6. *Assume that the data satisfies (42), namely:*

$$\frac{2\mu^3}{C_1^6\mathcal{P}_4^4} > \frac{2\mathcal{P}_1^2C_1^2}{\mu}||\mathbf{f}_1||^2_{L^2(\Omega_1)}+\frac{4\mathcal{P}_2^2}{\lambda_{\min}}||f_2||^2_{L^2(\Omega_2)}+4C_0^2\lambda_{\max}||g_{\mathrm{D}}||^2_{H^{1/2}(\Gamma_{2\mathrm{D}})}+\frac{4C_3^2}{\lambda_{\min}}||g_{\mathrm{N}}||^2_{L^2(\Gamma_{2\mathrm{N}})}.$$

Then problem $(\tilde{W}_B)$ has at most one weak solution satisfying

$$\|\boldsymbol{D}(\boldsymbol{u})\|_{L^2(\Omega_1)} \leq \frac{\mathcal{R}_0}{\sqrt{2\mu}}.$$

Proof. The proof is similar to the proof of uniqueness for the solution to (W_A) if we assume that the solution is bounded. □

A straightforward consequence due to Lemma 1 is the existence and uniqueness of a solution to problem (W_A) and the existence and local uniqueness to problem (W_B). In the next section, we propose a numerical scheme for solving the multiphysics problem that employs the continuous finite element method for the Navier-Stokes region with the discontinuous Galerkin method for the Darcy region.

3 A Multinumerics Scheme

Let $\mathcal{E}_1^h$ be a conforming triangulation of Ω_1 and let $\mathcal{E}_2^h$ be a general subdivision of Ω_2 consisting of triangular elements. The mesh $\mathcal{E}_2^h$ may contain hanging nodes. As usual, the parameter h denotes the maximum diameter of the elements. We assume that the resulting mesh $\mathcal{E}^h = \mathcal{E}_1^h \cup \mathcal{E}_2^h$ is regular [7]. In addition, we assume that the vertices of the polygonal line Γ_{12} are vertices in the mesh $\mathcal{E}^h$. However, the meshes $\mathcal{E}_1^h$ and $\mathcal{E}_2^h$ do not have to match on the interface Γ_{12}. In our numerical scheme, we propose to approximate the Navier-Stokes velocity and pressure in conforming finite element spaces $\boldsymbol{X}_1^h \subset \boldsymbol{X}_1$ and $M_1^h \subset M_1$ satisfying the discrete inf-sup condition with β_* independent of h:

$$\inf_{q_1\in M_1^h}\sup_{\boldsymbol{v}_1\in\boldsymbol{X}_1^h}\frac{|(\nabla\cdot\boldsymbol{v}_1, q_1)_{\Omega_1}|}{\|\nabla\boldsymbol{v}_1\|_{L^2(\Omega_1)}\|q_1\|_{L^2(\Omega_1)}} \geq \beta_* > 0. \tag{46}$$

Examples of such conforming finite elements are the Crouzeix-Raviart elements [8], the mini elements [3] and the Taylor-Hood elements [19]. We also propose to approximate the Darcy pressure in totally discontinuous finite element spaces. In order to define the discontinuous Galerkin method, we introduce further notation. We denote by Γ_2^h the set of interior edges in Ω_2. To each edge e of $\mathcal{E}_2^h$ we associate once and for all a unit normal vector $\boldsymbol{n}_e$. For $e \in \Gamma_{12}$, we set $\boldsymbol{n}_e = \boldsymbol{n}_{12}$, i.e. $\boldsymbol{n}_e$ is the exterior normal to Ω_1. If $\boldsymbol{n}_e$ points from the element E^1 to the element E^2, the jump $[]$ and average $\{\}$ of a function φ are given by:

$$[\varphi] = \varphi|_{E^1} - \varphi|_{E^2}, \quad \{\varphi\} = \frac{1}{2}\varphi|_{E^1} + \frac{1}{2}\varphi|_{E^2}.$$

For an integer $k_2 \geq 1$, we define

$$M_2^h = \{q_2 \in L^2(\Omega_2)\,;\, q_2|_{\Gamma_{2\mathrm{D}}} = 0 \quad \text{and} \quad \forall E \in \mathcal{E}_2^h, \quad q_2|_E \in I\!P_{k_2}(E)\},$$

equipped with the usual DG norm:

$$\forall q_2 \in M_2^h, \quad |||q_2||| = \Big(\sum_{E\in\mathcal{E}_2^h} ||\boldsymbol{K}^{1/2}\nabla q_2||^2_{L^2(E)} + \sum_{e\in\Gamma_2^h} \frac{1}{|e|}||[q_2]||^2_{L^2(e)} \Big)^{1/2}. \tag{47}$$

Lemma 7. *Assume that $p_{\mathrm{D}} \in H^{k_2+1}(\Omega_2)$ is the lift defined in (14)-(16). Then, there exists $P_{\mathrm{D}} \in M_2^h$ and a constant C independent of h satisfying:*

$$P_{\mathrm{D}} = 0, \quad on \quad \Gamma_{12}, \tag{48}$$

$$|||p_{\mathrm{D}} - P_{\mathrm{D}}||| \leq Ch^{k_2}\|p_{\mathrm{D}}\|_{H^{k_2+1}(\Omega_2)}. \tag{49}$$

In the rest of the text, we denote by C a generic constant independent of h and μ, that takes different values at different places. Next, we define several bilinear forms: $a_{\mathrm{NS}}, b_{\mathrm{NS}}, c_{\mathrm{NS}}$ are the discretizations of the viscous term, pressure term and nonlinear term respectively in the Navier-Stokes equations; a_{D} is the discretization of the diffusion term in the Darcy equations; and γ_α, $\alpha = A, B$, is the form containing terms related to the interface Γ_{12}.

$$\forall \boldsymbol{v}_1, \boldsymbol{w}_1 \in \boldsymbol{X}_1^h, \quad a_{\mathrm{NS}}(\boldsymbol{v}_1, \boldsymbol{w}_1) = 2\mu(\boldsymbol{D}(\boldsymbol{v}_1), \boldsymbol{D}(\boldsymbol{w}_1))_{\Omega_1},$$

$$\forall \boldsymbol{v}_1 \in \boldsymbol{X}_1^h, \forall q_1 \in M_1^h, \quad b_{\mathrm{NS}}(\boldsymbol{v}_1, q_1) = -(q_1, \nabla\cdot\boldsymbol{v}_1)_{\Omega_1},$$

$$\forall \boldsymbol{z}_1, \boldsymbol{v}_1, \boldsymbol{w}_1 \in \boldsymbol{X}_1^h, \quad c_{\mathrm{NS}}(\boldsymbol{z}_1, \boldsymbol{v}_1, \boldsymbol{w}_1) = \frac{1}{2}(\boldsymbol{z}_1\cdot\nabla\boldsymbol{v}_1, \boldsymbol{w}_1)_{\Omega_1} - \frac{1}{2}(\boldsymbol{z}_1\cdot\nabla\boldsymbol{w}_1, \boldsymbol{v}_1)_{\Omega_1} + \frac{1}{2}(\boldsymbol{z}_1\cdot\boldsymbol{n}_{12}, \boldsymbol{v}_1\cdot\boldsymbol{w}_1)_{\Gamma_{12}},$$

$$\forall q_2, t_2 \in M_2^h, \quad a_{\mathrm{D}}(q_2, t_2) = \sum_{E\in\mathcal{E}_2^h}\big(\boldsymbol{K}\nabla q_2, \nabla t_2\big)_E - \sum_{e\in\Gamma_2^h}\big(\{\boldsymbol{K}\nabla q_2\cdot\boldsymbol{n}_e\}[t_2]\big)_e$$
$$+\epsilon\sum_{e\in\Gamma_2^h}\big(\{\boldsymbol{K}\nabla t_2\cdot\boldsymbol{n}_e\}, [q_2]\big)_e + \sum_{e\in\Gamma_2^h}\frac{\sigma_e}{|e|}\big([q_2],[t_2]\big)_e$$

$$\forall \boldsymbol{v}_1, \boldsymbol{w}_1 \in \boldsymbol{X}_1^h, \forall q_2, t_2 \in M_2^h, \quad \gamma_B(\boldsymbol{v}_1, q_2; \boldsymbol{w}_1, t_2) = \big(q_2, \boldsymbol{w}_1\cdot\boldsymbol{n}_{12}\big)_{\Gamma_{12}} + \frac{1}{G}\big(\boldsymbol{v}_1\cdot\tau_{12}, \boldsymbol{w}_1\cdot\tau_{12}\big)_{\Gamma_{12}} - \big(\boldsymbol{v}_1\cdot\boldsymbol{n}_{12}, t_2\big)_{\Gamma_{12}},$$

$$\forall \boldsymbol{v}_1, \boldsymbol{w}_1 \in \boldsymbol{X}_1^h, \forall q_2, t_2 \in M_2^h, \quad \gamma_A(\boldsymbol{v}_1, q_2; \boldsymbol{w}_1, t_2) = \gamma_B(\boldsymbol{v}_1, q_2; \boldsymbol{w}_1, t_2) - \frac{1}{2}\big(\boldsymbol{v}_1\cdot\boldsymbol{v}_1, \boldsymbol{w}_1\cdot\boldsymbol{n}_{12}\big)_{\Gamma_{12}}.$$

In the definition of a_{D} the parameter ϵ yields a symmetric bilinear form if $\epsilon = -1$ and a non-symmetric bilinear form if $\epsilon = 0$ or $\epsilon = 1$. The parameter σ_e is a penalty parameter that varies with respect to the edge in $\mathcal{E}_2^h$. We recall that a_{D} is coercive and corresponds to the NIPG ($\epsilon = 1$), SIPG ($\epsilon = -1$) or IIPG ($\epsilon = 0$) methods [29, 17, 9]. There exists a constant $\kappa > 0$ independent of h such that:

$$\forall q_2 \in M_2^h, \quad \kappa|||q_2|||^2 \leq a_{\mathrm{D}}(q_2, q_2). \tag{50}$$

It has been shown that if $\epsilon \in \{-1, 0\}$, property (50) is valid if the penalty parameter is large enough. From [14], the lower bound for the penalty parameter is:

$$\forall e = \partial E_e^1 \cap \partial E_e^2, \quad \sigma_e \geq \frac{3\lambda_{\max}^2}{2\lambda_{\min}} k_2(k_2+1)(\cot\theta_{E_e^1} + \cot\theta_{E_e^2}),$$

where $\theta_{E_e^i}$ denotes the smallest angle in the triangle E_e^i. We also define the form L:

$$\begin{aligned}\forall \boldsymbol{v}_1 \in \boldsymbol{X}_1^h, \quad \forall q_2 \in M_2^h, \quad L(\boldsymbol{v}_1, q_2) \quad &= \quad \big(\boldsymbol{f}_1, \boldsymbol{v}_1\big)_{\Omega_1} + \big(f_2, q_2\big)_{\Omega_2} + \big(g_{\mathrm{N}}, q_2\big)_{\Gamma_{2\mathrm{N}}} \\ & \quad - \sum_{E \in \mathcal{E}_2^h} (\boldsymbol{K}\nabla p_{\mathrm{D}}, \nabla q_2)_E + \sum_{e \in \Gamma_2^h} (\{\boldsymbol{K}\nabla p_{\mathrm{D}} \cdot \boldsymbol{n}_e\}, [q_2])_e.\end{aligned}$$

We can now introduce the numerical solutions to problem (W_A) and to problem (W_B):

$$(W_A^h) \begin{cases} \text{ Find } \boldsymbol{U}_1 \in \boldsymbol{X}_1^h, P_1 \in M_1^h, P_2 = \Phi_2 + P_{\mathrm{D}}, \text{ with } \Phi_2 \in M_2^h, \text{ s.t.} \\ \forall \boldsymbol{v}_1 \in \boldsymbol{X}_1^h, \forall q_2 \in M_2^h, \;\; a_{\mathrm{NS}}(\boldsymbol{U}_1, \boldsymbol{v}_1) + b_{\mathrm{NS}}(\boldsymbol{v}_1, P_1) + c_{\mathrm{NS}}\big(\boldsymbol{U}_1; \boldsymbol{U}_1, \boldsymbol{v}_1\big) \\ +a_{\mathrm{D}}(\Phi_2, q_2) + \gamma_A(\boldsymbol{U}_1, \Phi_2; \boldsymbol{v}_1, q_2) = L(\boldsymbol{v}_1, q_2), \\ \forall q_1 \in M_1^h, \;\; b_{\mathrm{NS}}(\boldsymbol{U}_1, q_1) = 0. \end{cases}$$

$$(W_B^h) \begin{cases} \text{ Find } \boldsymbol{U}_1 \in \boldsymbol{X}_1^h, P_1 \in M_1^h, P_2 = \Phi_2 + P_{\mathrm{D}}, \text{ with } \Phi_2 \in M_2^h, \text{ s.t.} \\ \forall \boldsymbol{v}_1 \in \boldsymbol{X}_1^h, \forall q_2 \in M_2^h, \;\; a_{\mathrm{NS}}(\boldsymbol{U}_1, \boldsymbol{v}_1) + b_{\mathrm{NS}}(\boldsymbol{v}_1, P_1) + c_{\mathrm{NS}}\big(\boldsymbol{U}_1; \boldsymbol{U}_1, \boldsymbol{v}_1\big) \\ +a_{\mathrm{D}}(\Phi_2, q_2) + \gamma_B(\boldsymbol{U}_1, \Phi_2; \boldsymbol{v}_1, q_2) = L(\boldsymbol{v}_1, q_2), \\ \forall q_1 \in M_1^h, \;\; b_{\mathrm{NS}}(\boldsymbol{U}_1, q_1) = 0. \end{cases}$$

We end this section by giving important properties of the discrete spaces and the continuity property of the bilinear form c_{NS}.
Approximation properties. Assume that $(\boldsymbol{v}_1, p_1, p_2) \in \boldsymbol{X}_1 \times M_1 \times M_2$ is smooth enough, i.e. $\boldsymbol{v}_1 \in H^{k_1+1}(\Omega_1)$, $p_1 \in H^{k_1}(\Omega_1)$ and $p_2 \in H^{k_2+1}(\Omega_2)$ for integers k_1, k_2. Then, there exists an approximation $(\tilde{\boldsymbol{v}}_1, \tilde{p}_1, \tilde{p}_2) \in \boldsymbol{X}_1^h \times M_1^h \times M_2^h$ such that

$$\|\nabla(\boldsymbol{v}_1 - \tilde{\boldsymbol{v}}_1)\|_{L^2(\Omega_1)} \leq Ch^{k_1}\|\boldsymbol{v}_1\|_{H^{k_1+1}(\Omega_1)}, \tag{51}$$

$$\forall q_1 \in M_1^h, \quad (\nabla \cdot (\boldsymbol{v}_1 - \tilde{\boldsymbol{v}}_1), q_1)_{\Omega_1} = 0, \tag{52}$$

$$\|p_1 - \tilde{p}_1\|_{L^2(\Omega_1)} \leq Ch^{k_1}\|p_1\|_{H^{k_1}(\Omega_1)}, \tag{53}$$

$$i = 0, 1, \quad \|\nabla^i(p_2 - \tilde{p}_2)\|_{L^2(\Omega_2)} \leq Ch^{k_2+1-i}\|p_2\|_{H^{k_2+1}(\Omega_2)}. \tag{54}$$

It is easy to check that (54) implies

$$|||p_2 - \tilde{p_2}||| \leq Ch^{k_2}\|p_2\|_{H^{k_2+1}(\Omega_2)}. \tag{55}$$

L^2 bound. There exists a constant $C_5 > 0$ independent of h such that

$$\forall q_2 \in M_2^h, \quad \|q_2\|_{L^2(\Omega_2)} \leq C_5|||q_2|||. \tag{56}$$

Trace theorem. There exists a constant $C_6 > 0$ independent of h such that

$$\forall q_2 \in M_2^h, \quad \|q_2\|_{L^2(\Gamma_{12})} \leq C_6|||q_2|||. \tag{57}$$

The proof of (56) is given in Lemma 6.2 of [17] and the proof of (57) is given in Theorem 4.4 of [16]. We next show that the form c_{NS} is continuous.

Lemma 8. *There exists a constant C_7 such that*

$$\forall \boldsymbol{z}_1, \boldsymbol{v}_1, \boldsymbol{w}_1 \in \boldsymbol{X}_1, \quad c_{\mathrm{NS}}(\mathbf{z}_1; \mathbf{v}_1, \mathbf{w}_1) \leq C_7\|\boldsymbol{D}(\boldsymbol{z}_1)\|_{L^2(\Omega_1)}\|\boldsymbol{D}(\boldsymbol{v}_1)\|_{L^2(\Omega_1)}\|\boldsymbol{D}(\boldsymbol{w}_1)\|_{L^2(\Omega_1)}. \tag{58}$$

An expression for the constant C_7 is

$$C_7 = C_1^3(\mathcal{P}_4^2 + \frac{1}{2}C_2C_4^2).$$

Proof. Using (17), we have

$$\begin{aligned} c_{\rm NS}(\boldsymbol{z}_1;\boldsymbol{v}_1,\boldsymbol{w}_1) &= \frac{1}{2}(\boldsymbol{z}_1\cdot\nabla\boldsymbol{v}_1,\boldsymbol{w}_1)_{\Omega_1} - \frac{1}{2}(\boldsymbol{z}_1\cdot\nabla\boldsymbol{w}_1,\boldsymbol{v}_1)_{\Omega_1} + \frac{1}{2}(\boldsymbol{z}_1\cdot\boldsymbol{n}_{12},\boldsymbol{v}_1\cdot\boldsymbol{w}_1)_{\Gamma_{12}} \\ &\leq \frac{1}{2}||\boldsymbol{z}_1||_{L^4(\Omega_1)}\big(||\nabla\boldsymbol{v}_1||_{L^2(\Omega_1)}||\boldsymbol{w}_1||_{L^4(\Omega_1)} + ||\nabla\boldsymbol{w}_1||_{L^2(\Omega_1)}||\boldsymbol{v}_1||_{L^4(\Omega_1)}\big) \\ &\quad + \frac{1}{2}||\boldsymbol{z}_1||_{L^2(\Gamma_{12})}||\boldsymbol{v}_1||_{L^4(\Gamma_{12})}||\boldsymbol{w}_1||_{L^4(\Gamma_{12})}. \end{aligned}$$

Using (19), (21) and (20) we have

$$\begin{aligned} c_{\rm NS}(\boldsymbol{z}_1;\boldsymbol{v}_1,\boldsymbol{w}_1) &\leq (\mathcal{P}_4^2 + \frac{1}{2}C_2C_4^2)\|\nabla\boldsymbol{z}_1\|_{L^2(\Omega_1)}\|\nabla\boldsymbol{v}_1\|_{L^2(\Omega_1)}\|\nabla\boldsymbol{w}_1\|_{L^2(\Omega_1)} \\ &\leq C_1^3(\mathcal{P}_4^2 + \frac{1}{2}C_2C_4^2)\|\boldsymbol{D}(\boldsymbol{z}_1)\|_{L^2(\Omega_1)}\|\boldsymbol{D}(\boldsymbol{v}_1)\|_{L^2(\Omega_1)}\|\boldsymbol{D}(\boldsymbol{w}_1)\|_{L^2(\Omega_1)}. \end{aligned}$$

□

3.1 Consistency

Lemma 9. *Let $(\boldsymbol{u}_1, p_1, p_2)$ be the solution to (1)-(12) that is smooth enough. Define $\varphi_2 = p_2 - p_{\rm D}$. Then, we have for all $v_1 \in \mathbf{X}_1^h, q_2 \in \mathbf{M}_2^h, q_1 \in \mathbf{M}_1^h$:*

$$a_{\rm NS}(\boldsymbol{u}_1,\boldsymbol{v}_1) + b_{\rm NS}(\boldsymbol{v}_1,p_1) + c_{\rm NS}\big(\mathbf{u}_1;\mathbf{u}_1,\mathbf{v}_1\big) + a_{\rm D}(\varphi_2,q_2) + \gamma_A(\boldsymbol{u}_1,\varphi_2;\boldsymbol{v}_1,q_2) = L(\boldsymbol{v}_1,q_2), \tag{59}$$
$$b_{\rm NS}(\boldsymbol{u}_1,q_1) = 0. \tag{60}$$

Proof. Equation (60) is simply obtained by multiplying (2) by $q_1 \in M_1^h$ and integrating over Ω_1. Next, we multiply (1) by a test function $\boldsymbol{v}_1 \in \boldsymbol{X}_1^h$, integrate over Ω_1 and use Green's theorem. The resulting equation is exactly (25). Finally, we multiply (5) by a test function $q_2 \in M_2^h$, integrate over one element E, apply Green's theorem and sum over all elements in $\mathcal{E}_2^h$.

$$\sum_{E\in\mathcal{E}_2^h}(\boldsymbol{K}\nabla p_2,\nabla q_2)_E - \sum_{e\in\Gamma_2^h}(\{\boldsymbol{K}\nabla p_2\cdot\boldsymbol{n}_e\},[q_2])_e + \sum_{e\in\Gamma_{12}}(\boldsymbol{K}\nabla p_2\cdot\boldsymbol{n}_{12},q_2)_{\Gamma_{12}} = (f_2,q_2)_{\Omega_2} + (g_{\rm N},q_2)_{\Gamma_{12}}.$$

Using the splitting $p_2 = \varphi_2 + p_{\rm D}$, we obtain:

$$\sum_{E\in\mathcal{E}_2^h}(\boldsymbol{K}\nabla\varphi_2,\nabla q_2)_E - \sum_{e\in\Gamma_2^h}(\{\boldsymbol{K}\nabla\varphi_2\cdot\boldsymbol{n}_e\},[q_2])_e + \sum_{e\in\Gamma_{12}}(\boldsymbol{K}\nabla p_2\cdot\boldsymbol{n}_{12},q_2)_{\Gamma_{12}} \tag{61}$$
$$= (f_2,q_2)_{\Omega_2} + (g_{\rm N},q_2)_{\Gamma_{12}} - \sum_{E\in\mathcal{E}_2^h}(\boldsymbol{K}\nabla p_{\rm D},\nabla q_2)_E + \sum_{e\in\Gamma_2^h}(\{\boldsymbol{K}\nabla p_{\rm D}\cdot\boldsymbol{n}_e\},[q_2])_e. \tag{62}$$

We then add (25) and (62), and use the fact that $[\varphi_2]|_e = 0$ in $L^2(e)$ for all $e \in \Gamma_2^h$.

$$\begin{aligned} &2\mu(\boldsymbol{D}(\boldsymbol{u}_1),\boldsymbol{D}(\boldsymbol{v}_1))_{\Omega_1} - (p_1,\nabla\cdot\boldsymbol{v}_1)_{\Omega_1} + (\boldsymbol{u}_1\cdot\nabla\boldsymbol{u}_1,\boldsymbol{v}_1)_{\Omega_1} \\ &+ \sum_{E\in\mathcal{E}_2^h}(\boldsymbol{K}\nabla\varphi_2,\nabla q_2)_E - \sum_{e\in\Gamma_2^h}(\{\boldsymbol{K}\nabla\varphi_2\cdot\boldsymbol{n}_e\},[q_2])_e + \epsilon\sum_{e\in\Gamma_2^h}(\{\boldsymbol{K}\nabla q_2\cdot\boldsymbol{n}_e\},[\varphi_2])_e \\ &+ \sum_{e\in\Gamma_{12}}(\boldsymbol{K}\nabla p_2\cdot\boldsymbol{n}_{12},q_2)_{\Gamma_{12}} + ((-2\mu\boldsymbol{D}(\boldsymbol{u}_1) + p_1\boldsymbol{I})\boldsymbol{n}_{12},\boldsymbol{v}_1)_{\Gamma_{12}} \\ &= (\boldsymbol{f}_1,\boldsymbol{v}_1)_{\Omega_1} + (f_2,q_2)_{\Omega_2} + (g_{\rm N},q_2)_{\Gamma_{12}} - \sum_{E\in\mathcal{E}_2^h}(\boldsymbol{K}\nabla p_{\rm D},\nabla q_2)_E + \sum_{e\in\Gamma_2^h}(\{\boldsymbol{K}\nabla p_{\rm D}\cdot\boldsymbol{n}_e\},[q_2])_e. \end{aligned} \tag{63}$$

In this equation, the terms $\sum_{e\in\Gamma_{12}}(\boldsymbol{K}\nabla p_2\cdot\boldsymbol{n}_{12},q_2)_{\Gamma_{12}}+((-2\mu\boldsymbol{D}(\boldsymbol{u}_1)+p_1\boldsymbol{I})\boldsymbol{n}_{12},\boldsymbol{v}_1)_{\Gamma_{12}}$ are handled exactly as in the proof of Lemma 1. We remark that $\boldsymbol{u}_1\in\boldsymbol{V}_1$ and thus we have

$$(\boldsymbol{u}_1\cdot\nabla\boldsymbol{u}_1,\boldsymbol{v}_1)_{\Omega_1}=-(\boldsymbol{u}_1\cdot\nabla\boldsymbol{v}_1,\boldsymbol{u}_1)_{\Omega_1}+(\boldsymbol{u}_1\cdot\boldsymbol{n}_{12},\boldsymbol{v}_1\cdot\boldsymbol{u}_1)_{\Gamma_{12}},$$

which yields:

$$(\boldsymbol{u}_1\cdot\nabla\boldsymbol{u}_1,\boldsymbol{v}_1)_{\Omega_1}=c_{\rm NS}(\boldsymbol{u}_1,\boldsymbol{u}_1,\boldsymbol{v}_1).$$

Combining this result with (63), we obtain equation (59). □

Similarly, we can prove the following result:

Lemma 10. *Let $(\boldsymbol{u}_1,p_1,p_2)$ be the solution to (1)-(11) and (13) that is smooth enough. Define $\varphi_2=p_2-p_{\rm D}$. Then, we have for all $v_1\in\mathbf{X}_1^h,q_2\in\mathbf{M}_2^h,q_1\in\mathbf{M}_1^h$:*

$$a_{\rm NS}(\boldsymbol{u}_1,\boldsymbol{v}_1)+b_{\rm NS}(\boldsymbol{v}_1,p_1)+c_{\rm NS}\big(\mathbf{u}_1;\mathbf{u}_1,\mathbf{v}_1\big)+a_{\rm D}(\varphi_2,q_2)+\gamma_B(\boldsymbol{u}_1,\varphi_2;\boldsymbol{v}_1,q_2)=L(\boldsymbol{v}_1,q_2), \tag{64}$$

$$b_{\rm NS}(\boldsymbol{u}_1,q_1)=0. \tag{65}$$

3.2 Existence of Numerical Solution

We now proceed to show that there exists a unique solution to problem (W_A^h). We define the space of weakly divergence-free functions:

$$\boldsymbol{V}_1^h=\{\boldsymbol{v}_1\in\boldsymbol{X}_1^h:\forall q_1\in M_1^h,\quad b_{\rm NS}(\boldsymbol{v}_1,q_1)=0\}.$$

We note that $\boldsymbol{U}_1\in\boldsymbol{V}_1^h$ so that the scheme (W_A^h) reduces to:

$$\begin{aligned}\forall v_1\in\boldsymbol{X}_1^h,\forall q_2\in\boldsymbol{M}_2^h,\ &a_{\rm NS}(\boldsymbol{U}_1,\boldsymbol{v}_1)+b_{\rm NS}(\boldsymbol{v}_1,P_1)+c_{\rm NS}\big(\boldsymbol{U}_1;\boldsymbol{U}_1,\boldsymbol{v}_1\big)\\&+a_{\rm D}(P_2,q_2)+\gamma_A(\boldsymbol{U}_1,P_2;\boldsymbol{v}_1,q_2)=L(\boldsymbol{v}_1,q_2).\end{aligned} \tag{66}$$

Clearly, if $(\boldsymbol{U}_1,P_1,P_2)$ is a solution to (W_A^h), then $(\boldsymbol{U}_1,P_2)$ is a solution to (66). Conversely, assume that $(\boldsymbol{U}_1,P_2)$ is a solution to (66). Then, the discrete inf-sup (46) implies that there exists a unique $P_1\in M_1^h$ such that $(\boldsymbol{U}_1,P_1,P_2)$ is a solution to (W_A^h). Based on this equivalence between the two problems, it suffices to show that there exists a solution $(\boldsymbol{U}_1,P_2)\in\boldsymbol{V}_1^h\times\boldsymbol{M}_2^h$ of (66). We will use Lemma 2 and we define the inner-product on $Y^h=\boldsymbol{V}_1^h\times M_2^h$:

$$\big((\boldsymbol{v}_1,q_2),(\boldsymbol{w}_1,t_2)\big)_{Y^h}=2\mu\big(\boldsymbol{D}(\boldsymbol{v}_1),\boldsymbol{D}(\boldsymbol{w}_1)\big)_{\Omega_1}+\sum_{E\in\mathcal{E}_h^2}\big(\boldsymbol{K}\nabla q_2,\nabla t_2\big)_E+\sum_{e\in\Gamma_2^h}\frac{1}{|e|}\big([q_2],[t_2]\big)_e. \tag{67}$$

Next define $\Psi_A^h:Y^h\to Y^h$ such that:

$$\big(\Psi_A^h(\boldsymbol{v}_1,q_2),(\boldsymbol{w}_1,t_2)\big)_{Y^h}\quad=\quad a_{\rm NS}(\boldsymbol{v}_1,\boldsymbol{w}_1)+c_{\rm NS}\big(\boldsymbol{v}_1;\boldsymbol{v}_1,\boldsymbol{w}_1\big)+a_{\rm D}(q_2,t_2)+\gamma_A(\boldsymbol{v}_1,q_2;\boldsymbol{w}_1,t_2)-L(\boldsymbol{w}_1,t_2).$$

Using (50) and the definitions of the bilinear forms, we obtain a lower bound of $\big(\Psi_A^h(\boldsymbol{v}_1,q_2),(\boldsymbol{v}_1,q_2)\big)_{Y^h}$:

$$\big(\Psi_A^h(\boldsymbol{v}_1,q_2),(\boldsymbol{v}_1,q_2)\big)_{Y^h}\geq 2\mu\|\boldsymbol{D}(\boldsymbol{v}_1)\|^2_{L^2(\Omega_1)}+\kappa|||q_2|||^2+\frac{1}{G}\|\boldsymbol{v}_1\cdot\boldsymbol{\tau}_{12}\|^2_{L^2(\Gamma_{12})}-L(\boldsymbol{v}_1,q_2).$$

From (17), (18) and (56), we have for any $\delta>0$:

$$\big(f_2,q_2)_{\Omega_2}\leq\frac{\delta}{2}|||q_2|||^2+\frac{C_5^2}{2\delta}||f_2||^2_{L^2(\Omega_2)}. \tag{68}$$

Similarly, from (17), (18) and (57), we have for any $\delta>0$:

$$(g_{\rm N},q_2)_{\Gamma_{12}}\leq\frac{\delta}{2}|||q_2|||^2+\frac{C_6^2}{2\delta}||g_{\rm N}||^2_{L^2(\Gamma_{12})}. \tag{69}$$

Using a trace theorem [26], (17), (18) and (24), we have for any $\delta > 0$:

$$\big| -\sum_{E\in\mathcal{E}_2^h}(\boldsymbol{K}\nabla p_{\rm D},\nabla q_2)_E + \sum_{e\in\Gamma_2^h}(\{\boldsymbol{K}\nabla p_{\rm D}\cdot\boldsymbol{n}_e\},[q_2])_e\big| \le \delta|||q_2|||^2 + \frac{\lambda_{\max}}{2\delta}\|p_{\rm D}\|^2_{H^1(\Omega_2)} + \frac{C_t^2}{2\delta}\sum_{E\in\mathcal{E}_2^h}\|p_{\rm D}\|^2_{H^2(E)}. \quad (70)$$

Combining the bounds (68), (69), (70) and (34), we obtain:

$$\big(\Psi_A^h(\boldsymbol{v}_1,q_2),(\boldsymbol{v}_1,q_2)\big)_{Y^h} \ge \frac{3\mu}{2}\|\boldsymbol{D}(\boldsymbol{v}_1)\|^2_{L^2(\Omega_1)} + \frac{\kappa}{2}|||q_2|||^2 + \frac{1}{G}\|\boldsymbol{v}_1\cdot\boldsymbol{\tau}_{12}\|^2_{L^2(\Gamma_{12})}$$
$$-\Big(\frac{\mathcal{P}_1^2C_1^2}{2\mu}\|\boldsymbol{f}_1\|^2_{L^2(\Omega_1)} + \frac{2C_5^2}{\kappa}\|f_2\|^2_{L^2(\Omega_2)} + \frac{2C_6^2}{\kappa}\|g_{\rm N}\|^2_{L^2(\Gamma_{2\rm N})} + \frac{2\lambda_{\max}}{\kappa}\|p_{\rm D}\|^2_{H^1(\Omega_1)} + \frac{2C_t^2}{\kappa}\sum_{E\in\mathcal{E}_2^h}\|p_{\rm D}\|^2_{H^2(E)}\Big).$$

Therefore, $\big(\Psi_A^h(\boldsymbol{v}_1,q_2),(\boldsymbol{v}_1,q_2)\big)_{Y^h} \ge 0$ provided that $||(\boldsymbol{v}_1,q_2)||_{Y^h} = \mathcal{R}_2$ with

$$\mathcal{R}_2 = (\max(\frac{3}{4},\frac{\kappa}{2}))^{1/2}\big(\frac{\mathcal{P}_1^2C_1^2}{2\mu}\|\boldsymbol{f}_1\|^2_{L^2(\Omega_1)} + \frac{2C_5^2}{\kappa}\|f_2\|^2_{L^2(\Omega_2)} + \frac{2C_6^2}{\kappa}\|g_{\rm N}\|^2_{L^2(\Gamma_{2\rm N})} + \frac{2\lambda_{\max}}{\kappa}\|p_{\rm D}\|^2_{H^1(\Omega_1)} \quad (71)$$
$$+\frac{2C_t^2}{\kappa}\sum_{E\in\mathcal{E}_2^h}\|p_{\rm D}\|^2_{H^2(E)}\big)^{1/2}. \quad (72)$$

This concludes the proof of existence of a solution $(\boldsymbol{U}_1,P_2)$ of (66). The same argument can be used to show that any solution $(\boldsymbol{U}_1,P_2)$ of (66) is bounded as follows:

$$2\mu\|\boldsymbol{D}(\boldsymbol{U}_1)\|^2_{L^2(\Omega_1)} + |||P_2|||^2 \le \mathcal{R}_2^2. \quad (73)$$

The proof of existence of a solution to (W_B^h) is more technical as the nonlinear term $\frac{1}{2}(\boldsymbol{z}_1\cdot\boldsymbol{n}_{12},\boldsymbol{v}_1\cdot\boldsymbol{w}_1)_{\Gamma_{12}}$ of the form $c_{\rm NS}$ remains. As above, we define $\Psi_B^h : Y^h \to Y^h$ such that:

$$\big(\Psi_B^h(\boldsymbol{v}_1,q_2),(\boldsymbol{w}_1,t_2)\big)_{Y^h} = a_{\rm NS}(\boldsymbol{v}_1,\boldsymbol{w}_1) + c_{\rm NS}\big(\boldsymbol{v}_1;\boldsymbol{v}_1,\boldsymbol{w}_1\big) + a_{\rm D}(q_2,t_2) + \gamma_B(\boldsymbol{v}_1,q_2;\boldsymbol{w}_1,t_2) - L(\boldsymbol{w}_1,t_2).$$

This yields

$$\big(\Psi_B^h(\boldsymbol{v}_1,q_2),(\boldsymbol{v}_1,q_2)\big)_{Y^h} \ge 2\mu\|\boldsymbol{D}(\boldsymbol{v}_1)\|^2_{L^2(\Omega_1)} + \kappa|||q_2|||^2 + \frac{1}{G}\|\boldsymbol{v}_1\cdot\boldsymbol{\tau}_{12}\|^2_{L^2(\Gamma_{12})} + \frac{1}{2}(\boldsymbol{v}_1\cdot\boldsymbol{n}_{12},\boldsymbol{v}_1\cdot\boldsymbol{v}_1)_{\Gamma_{12}} - L(\boldsymbol{v}_1,q_2).$$

Using the bound

$$\frac{1}{2}(\boldsymbol{v}_1\cdot\boldsymbol{n}_{12},\boldsymbol{v}_1\cdot\boldsymbol{v}_1)_{\Gamma_{12}} \le \frac{C_1^3C_2C_4^2}{2}\|\boldsymbol{D}(\boldsymbol{v}_1)\|^3_{L^2(\Omega_1)}$$

and the bounds (68), (69), (70) and (34), we obtain that $\big(\Psi_B^h(\boldsymbol{v}_1,q_2),(\boldsymbol{v}_1,q_2)\big)_{Y^h} \ge 0$ provided that $||(\boldsymbol{v}_1,q_2)||_{Y^h} = \mathcal{R}_2$ and that

$$2\mu\|\boldsymbol{D}(\boldsymbol{v}_1)\|^2_{L^2(\Omega_1)} \le \frac{32\mu^3}{C_1^6C_2^2C_4^4}.$$

These conditions are compatible if we assume that

$$\mathcal{R}_2^2 < \frac{32\mu^3}{C_1^6C_2^2C_4^4}.$$

The rest of the proof is similar. We summarize our results in the following theorem.

Theorem 11. *Let $\mathcal{R}_2$ be defined by (72). There exists a solution $(\boldsymbol{U}_1,P_1,P_2)$ of (W_A^h) satisfying (73). If the data satisfies*

$$\mathcal{R}_2^2 < \frac{32\mu^3}{C_1^6C_2^2C_4^4}, \quad (74)$$

then there exists a solution $(\boldsymbol{U}_1,P_1,P_2)$ of (W_B^h) satisfying (73).

3.3 Uniqueness of Numerical Solution

Theorem 12. *Let $\mathcal{R}_2$ be defined by (72). Under the condition*

$$\mu^{3/2} > \frac{C_1^3}{\sqrt{2}}(\mathcal{P}_4^2 + C_2 C_4^2)\mathcal{R}_2 \tag{75}$$

problem (W_A^h) admits a unique solution.

Proof. It suffices to prove uniqueness of the solution to (66). We assume that $(\boldsymbol{U}_1^1, P_2^1)$ and $(\boldsymbol{U}_1^2, P_2^2)$ are two solutions satisfying (66), and let $\boldsymbol{W}_1 = \boldsymbol{U}_1^1 - \boldsymbol{U}_1^2$ and $\chi_2 = P_2^1 - P_2^2$.

$$a_{\rm NS}(\boldsymbol{W}_1, \boldsymbol{v}_1) + c_{\rm NS}(\boldsymbol{U}_1^1, \boldsymbol{U}_1^1, \boldsymbol{v}_1) - c_{\rm NS}(\boldsymbol{U}_1^2, \boldsymbol{U}_1^2, \boldsymbol{v}_1) + a_D(\chi_2, q_2) + \big(\chi_2, \boldsymbol{v}_1 \cdot \boldsymbol{n}_{12}\big)_{\Gamma_{12}} - \frac{1}{2}\big(\boldsymbol{U}_1^1 \cdot \boldsymbol{U}_1^1, \boldsymbol{v}_1 \cdot n_{12}\big)_{\Gamma_{12}}$$
$$+\frac{1}{2}\big(\boldsymbol{U}_1^2 \cdot \boldsymbol{U}_1^2, \boldsymbol{v}_1 \cdot \boldsymbol{n}_{12}\big)_{\Gamma_{12}} + \frac{1}{G}\big(\boldsymbol{W}_1 \cdot \tau_{12}, \boldsymbol{v}_1 \cdot \tau_{12}\big)_{\Gamma_{12}} - \big(\boldsymbol{W}_1 \cdot \boldsymbol{n}_{12}, q_2\big)_{\Gamma_{12}} = 0.$$

In particular, we choose $\boldsymbol{v}_1 = \boldsymbol{W}_1$ and $q_2 = \chi_2$.

$$a_{NS}(\boldsymbol{W}_1, \boldsymbol{W}_1) + a_D(\chi_2, \chi_2) + \frac{1}{G}||\boldsymbol{W}_1 \cdot \tau_{12}||^2_{L^2(\Gamma_{12})} + c_{NS}\big(\boldsymbol{U}_1^1, \boldsymbol{U}_1^1, \boldsymbol{W}_1\big) - c_{NS}\big(\boldsymbol{U}_1^2, \boldsymbol{U}_1^2, \boldsymbol{W}_1\big) + \big(\chi_2, \boldsymbol{W}_1 \cdot \boldsymbol{n}_{12}\big)_{\Gamma_{12}}$$
$$-\frac{1}{2}\big(\boldsymbol{U}_1^1 \cdot \boldsymbol{U}_1^1, \boldsymbol{W}_1 \cdot \boldsymbol{n}_{12}\big)_{\Gamma_{12}} + \frac{1}{2}\big(\boldsymbol{U}_1^2 \cdot \boldsymbol{U}_1^2, \boldsymbol{W}_1 \cdot \boldsymbol{n}_{12}\big)_{\Gamma_{12}} - \big(\boldsymbol{W}_1 \cdot \boldsymbol{n}_{12}, \chi_2\big)_{\Gamma_{12}} = 0.$$

Using (50) and rewriting the nonlinear terms as

$$c_{\rm NS}(\boldsymbol{U}_1^1, \boldsymbol{U}_1^1, \boldsymbol{W}_1) - c_{\rm NS}(\boldsymbol{U}_1^2, \boldsymbol{U}_1^2, \boldsymbol{W}_1) = c_{\rm NS}(\boldsymbol{W}_1, \boldsymbol{U}_1^1, \boldsymbol{W}_1) + c_{\rm NS}(\boldsymbol{U}_1^2, \boldsymbol{W}_1, \boldsymbol{W}_1),$$
$$-\frac{1}{2}\big(\boldsymbol{U}_1^1 \cdot \boldsymbol{U}_1^1, \boldsymbol{W}_1 \cdot \boldsymbol{n}_{12}\big)_{\Gamma_{12}} + \frac{1}{2}\big(\boldsymbol{U}_1^2 \cdot \boldsymbol{U}_1^2, \boldsymbol{W}_1 \cdot \boldsymbol{n}_{12}\big)_{\Gamma_{12}} = -\frac{1}{2}(\boldsymbol{W}_1 \cdot \boldsymbol{U}_1^1, \boldsymbol{W}_1 \cdot \boldsymbol{n}_{12})_{\Gamma_{12}} - \frac{1}{2}(\boldsymbol{W}_1 \cdot \boldsymbol{U}_1^2, \boldsymbol{W}_1 \cdot \boldsymbol{n}_{12})_{\Gamma_{12}},$$

we obtain

$$2\mu\|\boldsymbol{D}(\boldsymbol{W}_1)\|^2_{L^2(\Omega_1)} + \kappa|||\chi_2|||^2 + \frac{1}{G}||\boldsymbol{W}_1 \cdot \tau_{12}||^2_{L^2(\Gamma_{12})}$$
$$+c_{\rm NS}(\boldsymbol{W}_1, \boldsymbol{U}_1^1, \boldsymbol{W}_1) + c_{\rm NS}(\boldsymbol{U}_1^2, \boldsymbol{W}_1, \boldsymbol{W}_1) - \frac{1}{2}(\boldsymbol{W}_1 \cdot \boldsymbol{U}_1^1, \boldsymbol{W}_1 \cdot \boldsymbol{n}_{12})_{\Gamma_{12}} - \frac{1}{2}(\boldsymbol{W}_1 \cdot \boldsymbol{U}_1^2, \boldsymbol{W}_1 \cdot \boldsymbol{n}_{12})_{\Gamma_{12}} \le 0.$$

From Lemma 8, we have

$$c_{\rm NS}(\boldsymbol{W}_1; \boldsymbol{U}_1^1, \boldsymbol{W}_1) + c_{\rm NS}(\boldsymbol{U}_1^2; \boldsymbol{W}_1, \boldsymbol{W}_1) \le C_7\|\boldsymbol{D}(\boldsymbol{W}_1)\|^2_{L^2(\Omega_1)}(\|\boldsymbol{D}(\boldsymbol{U}_1^1)\|_{L^2(\Omega_1)} + \|\boldsymbol{D}(\boldsymbol{U}_1^2)\|_{L^2(\Omega_1)}).$$

Similarly, using (21) and (20), we have

$$\frac{1}{2}(\boldsymbol{W}_1 \cdot \boldsymbol{U}_1^1, \boldsymbol{W}_1 \cdot \boldsymbol{n}_{12})_{\Gamma_{12}} + \frac{1}{2}(\boldsymbol{W}_1 \cdot \boldsymbol{U}_1^2, \boldsymbol{W}_1 \cdot \boldsymbol{n}_{12})_{\Gamma_{12}} \le \frac{1}{2}C_4^2 C_2 C_1^3\|\boldsymbol{D}(\boldsymbol{W}_1)\|^2_{L^2(\Omega_1)}(\|\boldsymbol{D}(\boldsymbol{U}_1^1)\|_{L^2(\Omega_1)} + \|\boldsymbol{D}(\boldsymbol{U}_1^2)\|_{L^2(\Omega_1)}).$$

Combining the two bounds above with (73), we obtain:

$$\big(2\mu - \frac{\mathcal{R}_2}{\sqrt{\mu}}(\sqrt{2}C_7 + \frac{1}{\sqrt{2}}C_1^3 C_2 C_4^2)\big)\|\boldsymbol{D}(\boldsymbol{W}_1)\|^2_{L^2(\Omega_1)} + \kappa|||\chi_2|||^2 + \frac{1}{G}||\boldsymbol{W}_1 \cdot \tau_{12}||^2_{L^2(\Gamma_{12})} \le 0.$$

This clearly implies that $\boldsymbol{W}_1 = \boldsymbol{0}$ and $\chi_2 = 0$ if the condition

$$2\mu > \frac{\mathcal{R}_2}{\sqrt{\mu}}(\sqrt{2}C_7 + \frac{1}{\sqrt{2}}C_1^3 C_2 C_4^2)$$

is satisfied. This condition is equivalent to (75). □

The proof of uniqueness for the solution to (W_B^h) involves less terms but is only valid in a certain ball. We skip the proof and state the result.

Theorem 13. *Let $\mathcal{R}_2$ be defined by (72). Under the condition (74) and the condition*

$$\mu^{3/2} > \frac{C_1^3}{\sqrt{2}}(\mathcal{P}_4^2 + \frac{1}{2}C_2C_4^2)\mathcal{R}_2 \tag{76}$$

problem (W_B^h) admits at most one solution satisfying (73).

4 A Priori Error Estimates

Theorem 14. *Assume that the solution to problem (W_A) is smooth enough, i.e. $\boldsymbol{u}_1 \in (H^{k_1+1}(\Omega_1))^2, p_1 \in H^{k_1}(\Omega_1)$ and $p_2 = \varphi_2 + p_{\mathrm{D}}$ with $\varphi_2 \in H^{k_2+1}(\Omega_2)$. Let $\mathcal{R}_1$ be defined by (41) and let $\mathcal{R}_2$ be defined by (72). Assume that the data satisfies:*

$$\mu^{3/2} > \frac{C_1^3}{\sqrt{2}}(\mathcal{P}_4^2 + C_2C_4^2)(\mathcal{R}_1 + \mathcal{R}_2).$$

Let $(\boldsymbol{U}_1, P_1, P_2)$ be a solution to problem (W_A^h). Then, there exists a constant C independent of h and μ such that

$$\mu\|\boldsymbol{D}(\boldsymbol{u}_1 - \boldsymbol{U}_1)\|_{L^2(\Omega_1)}^2 + |||\varphi_2 - \Phi_2|||^2 + \|(\boldsymbol{u}_1 - \boldsymbol{U}_1)\cdot\boldsymbol{\tau}_{12}\|_{L^2(\Gamma_{12})}^2 \le C\big(1 + \frac{(\mathcal{R}_1+\mathcal{R}_2)^2}{\mu^2}\big)h^{2k_1}\|\boldsymbol{u}_1\|_{H^{k_1+1}(\Omega_1)}^2$$
$$+C\big(1+\frac{1}{\mu}\big)h^{2k_2}\|\varphi_2\|_{H^{k_2+1}(\Omega_2)}^2 + C\frac{1}{\mu}h^{2k_1}\|p_1\|_{H^{k_1}(\Omega_1)}^2.$$

Proof. Let $\tilde{\boldsymbol{u}}_1, \tilde{p}_1, \tilde{\varphi}_2$ be approximations to $\boldsymbol{u}_1, p_1, \varphi_2$ in the spaces $\boldsymbol{X}_1^h, M_1^h$ and M_2^h respectively. Assume that the error bounds (51), (53) and (54) hold. Let

$$\boldsymbol{\chi}_1 = \boldsymbol{U}_1 - \tilde{\boldsymbol{u}}_1, \quad \xi_1 = P_1 - \tilde{p}_1, \quad \xi_2 = \Phi_2 - \tilde{\varphi}_2,$$
$$\boldsymbol{\zeta}_1 = \boldsymbol{u}_1 - \tilde{\boldsymbol{u}}_1, \quad \eta_1 = p_1 - \tilde{p}_1, \quad \eta_2 = \varphi_2 - \tilde{\varphi}_2.$$

Using the consistency Lemma 9 and the definition of problem (W_A^h), we obtain the error equations:

$$\forall \boldsymbol{v}_1 \in \boldsymbol{X}_1^h, \forall q_2 \in M_2^h, \quad a_{\mathrm{NS}}(\boldsymbol{\chi}_1, \boldsymbol{v}_1) + a_{\mathrm{D}}(\xi_2, q_2) + b_{\mathrm{NS}}(\boldsymbol{v}_1, \xi_1) + c_{\mathrm{NS}}(\boldsymbol{U}_1; \boldsymbol{U}_1, \boldsymbol{v}_1) - c_{\mathrm{NS}}(\boldsymbol{u}_1; \boldsymbol{u}_1, \boldsymbol{v}_1)$$
$$+\gamma_A(\boldsymbol{U}_1, \Phi_2; \boldsymbol{v}_1, q_2) - \gamma_A(\boldsymbol{u}_1, \varphi_2; \boldsymbol{v}_1, q_2) = a_{\mathrm{NS}}(\boldsymbol{\zeta}_1, \boldsymbol{v}_1) + a_{\mathrm{D}}(\eta_2, q_2) + b_{\mathrm{NS}}(\boldsymbol{v}_1, \eta_1),$$
$$\forall q_1 \in M_1^h, \quad b_{\mathrm{NS}}(\boldsymbol{\chi}_1, q_1) = b_{\mathrm{NS}}(\boldsymbol{\zeta}_1, q_1).$$

Let $\boldsymbol{v}_1 = \boldsymbol{\chi}_1, q_1 = \xi_1, q_2 = \xi_2$, then from (50), we have

$$2\mu\|\boldsymbol{D}(\boldsymbol{\chi}_1)\|_{L^2(\Omega_1)}^2 + \kappa|||\xi_2|||^2 + c_{\mathrm{NS}}(\boldsymbol{U}_1; \boldsymbol{U}_1, \boldsymbol{\chi}_1) - c_{\mathrm{NS}}(\boldsymbol{u}_1; \boldsymbol{u}_1, \boldsymbol{\chi}_1)$$
$$+\gamma_A(\boldsymbol{U}_1, \Phi_2; \boldsymbol{\chi}_1, \xi_2) - \gamma_A(\boldsymbol{u}_1, \varphi_2; \boldsymbol{\chi}_1, \xi_2) \le a_{\mathrm{NS}}(\boldsymbol{\zeta}_1, \boldsymbol{\chi}_1) + a_{\mathrm{D}}(\eta_2, \xi_2) + b_{\mathrm{NS}}(\boldsymbol{\chi}_1, \eta_1) - b_{\mathrm{NS}}(\boldsymbol{\zeta}_1, \xi_1). \tag{77}$$

We first expand the terms involving the linear form γ_A:

$$\gamma_A(\boldsymbol{U}_1, \Phi_2; \boldsymbol{\chi}_1, \xi_2) - \gamma_A(\boldsymbol{u}_1, \varphi_2; \boldsymbol{\chi}_1, \xi_2) = -\frac{1}{2}(\boldsymbol{U}_1\cdot\boldsymbol{U}_1, \boldsymbol{\chi}_1\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} + \frac{1}{2}(\boldsymbol{u}_1\cdot\boldsymbol{u}_1, \boldsymbol{\chi}_1\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} + \frac{1}{G}\|\boldsymbol{\chi}_1\cdot\boldsymbol{\tau}_{12}\|_{L^2(\Gamma_{12})}^2$$
$$-\frac{1}{G}(\boldsymbol{\zeta}_1\cdot\boldsymbol{\tau}_{12}, \boldsymbol{\chi}_1\cdot\boldsymbol{\tau}_{12})_{\Gamma_{12}} - (\eta_2, \boldsymbol{\chi}_1\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} + (\xi_2, \boldsymbol{\zeta}_1\cdot\boldsymbol{n}_{12})_{\Gamma_{12}}. \tag{78}$$

The nonlinear terms are rewritten as

$$A_1 = -\frac{1}{2}(\boldsymbol{U}_1\cdot\boldsymbol{U}_1, \boldsymbol{\chi}_1\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} + \frac{1}{2}(\boldsymbol{u}_1\cdot\boldsymbol{u}_1, \boldsymbol{\chi}_1\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} = \frac{1}{2}(\boldsymbol{U}_1\cdot\boldsymbol{\chi}_1, \boldsymbol{\chi}_1\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} + \frac{1}{2}(\boldsymbol{\chi}_1\cdot\boldsymbol{u}_1, \boldsymbol{\chi}_1\cdot\boldsymbol{n}_{12})_{\Gamma_{12}}$$
$$-\frac{1}{2}(\boldsymbol{U}_1\cdot\boldsymbol{\zeta}_1, \boldsymbol{\chi}_1\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} - \frac{1}{2}(\boldsymbol{\zeta}_1\cdot\boldsymbol{u}_1, \boldsymbol{\chi}_1\cdot\boldsymbol{n}_{12})_{\Gamma_{12}}, \tag{79}$$

and bounded by using (17), (18), (21), (20), (40) and (73)

$$\begin{aligned} A_1 \;\leq\;& \frac{1}{2}C_1^3C_2C_4^2\|\boldsymbol{D}(\boldsymbol{\chi}_1)\|^2_{L^2(\Omega_1)}(\|\boldsymbol{D}(\boldsymbol{U}_1)\|_{L^2(\Omega_1)} + \|\boldsymbol{D}(\boldsymbol{u}_1)\|_{L^2(\Omega_1)}) \\ &+C\|\boldsymbol{D}(\boldsymbol{\chi}_1)\|_{L^2(\Omega_1)}\|\nabla\boldsymbol{\zeta}_1\|_{L^2(\Omega_1)}(\|\boldsymbol{D}(\boldsymbol{U}_1)\|_{L^2(\Omega_1)} + \|\boldsymbol{D}(\boldsymbol{u}_1)\|_{L^2(\Omega_1)}) \\ \leq\;& \frac{\mu}{5}\|\boldsymbol{D}(\boldsymbol{\chi}_1)\|^2_{L^2(\Omega_1)} + \frac{1}{2}C_1^3C_2C_4^2\frac{\mathcal{R}_1+\mathcal{R}_2}{\sqrt{2\mu}}\|\boldsymbol{D}(\boldsymbol{\chi}_1)\|^2_{L^2(\Omega_1)} \\ &+\frac{C(\mathcal{R}_1+\mathcal{R}_2)^2}{\mu^2}\|\nabla\boldsymbol{\zeta}_1\|^2_{L^2(\Omega_1)}. \end{aligned}$$

The linear terms in (78) are bounded by (17), (18), (21), (20) and (57)

$$\frac{1}{G}(\boldsymbol{\zeta}_1\cdot\boldsymbol{\tau}_{12}, \boldsymbol{\chi}_1\cdot\boldsymbol{\tau}_{12})_{\Gamma_{12}} \leq \frac{1}{2G}||\boldsymbol{\chi}_1\cdot\boldsymbol{\tau}_{12}||^2_{L^2(\Gamma_{12})} + C\|\nabla\boldsymbol{\zeta}_1\|^2_{L^2(\Omega_1)}$$
$$(\eta_2, \boldsymbol{\chi}_1\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} \leq \frac{\mu}{5}\|\boldsymbol{D}(\boldsymbol{\chi}_1)\|^2_{L^2(\Omega_1)} + \frac{C}{\mu}|||\eta_2|||^2,$$
$$(\xi_2, \boldsymbol{\zeta}_1\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} \leq \frac{\kappa}{5}|||\xi_2|||^2 + C\|\nabla\boldsymbol{\zeta}_1\|^2_{L^2(\Omega_1)}.$$

We rewrite the nonlinear terms involving c_{NS} in (77) in a similar way as with the term A_1 defined in (79). We obtain a bound by using Lemma 8.

$$\begin{aligned} c_{\mathrm{NS}}(\boldsymbol{U}_1;\boldsymbol{U}_1,\boldsymbol{\chi}_1) - c_{\mathrm{NS}}(\boldsymbol{u}_1;\boldsymbol{u}_1,\boldsymbol{\chi}_1) \;=\;& c_{\mathrm{NS}}(\boldsymbol{U}_1;\boldsymbol{\chi}_1,\boldsymbol{\chi}_1) + c_{\mathrm{NS}}(\boldsymbol{\chi}_1;\boldsymbol{u}_1,\boldsymbol{\chi}_1) \\ &-c_{\mathrm{NS}}(\boldsymbol{U}_1;\boldsymbol{\zeta}_1,\boldsymbol{\chi}_1) - c_{\mathrm{NS}}(\boldsymbol{\zeta}_1;\boldsymbol{u},\boldsymbol{\chi}_1) \\ \leq\;& \frac{\mu}{5}\|\boldsymbol{D}(\boldsymbol{\chi}_1)\|^2_{L^2(\Omega_1)} + C_7\frac{\mathcal{R}_1+\mathcal{R}_2}{\sqrt{2\mu}}\|\boldsymbol{D}(\boldsymbol{\chi}_1)\|^2_{L^2(\Omega_1)} \\ &+C\frac{(\mathcal{R}_1+\mathcal{R}_2)^2}{\mu^2}\|\nabla\boldsymbol{\zeta}_1\|^2_{L^2(\Omega_1)}. \end{aligned}$$

The term $a_{\mathrm{NS}}(\boldsymbol{\zeta}_1,\boldsymbol{\chi}_1)$ is simply bounded using Cauchy-Schwarz and Young's inequalities.

$$a_{\mathrm{NS}}(\boldsymbol{\zeta}_1,\boldsymbol{\chi}_1) \leq \frac{\mu}{5}\|\boldsymbol{D}(\boldsymbol{\chi}_1)\|^2_{L^2(\Omega_1)} + C\mu\|\boldsymbol{D}(\boldsymbol{\zeta}_1)\|^2_{L^2(\Omega_1)}.$$

The term $b_{\mathrm{NS}}(\boldsymbol{\zeta}_1,\xi_1)$ vanishes because of property (52). The term $a_{\mathrm{D}}(\eta_2,\xi_2)$ is bounded using standard DG techniques (see [26]) and the approximation property (54):

$$a_{\mathrm{D}}(\eta_2,\xi_2) \leq \frac{\kappa}{4}|||\xi_2|||^2 + Ch^{2k_2}\|\varphi_2\|^2_{H^{k_2+1}(\Omega_2)}.$$

Finally, the term $b_{\mathrm{NS}}(\boldsymbol{\chi}_1,\eta_1)$ is bounded as:

$$b_{\mathrm{NS}}(\boldsymbol{\chi}_1,\eta_1) \leq \frac{\mu}{5}\|\boldsymbol{D}(\boldsymbol{\chi}_1)\|^2_{L^2(\Omega_1)} + \frac{C}{\mu}\|\eta_1\|^2_{L^2(\Omega_1)}.$$

Combining the results above, the error equation (77) becomes:

$$\Big(\mu - (\frac{1}{2}C_1^3C_2C_4^2 + C_7)\frac{\mathcal{R}_1+\mathcal{R}_2}{\sqrt{2\mu}}\Big)\|\boldsymbol{D}(\boldsymbol{\chi}_1)\|^2_{L^2(\Omega_1)} + \frac{\kappa}{2}|||\xi_2|||^2 + \frac{1}{2G}\|\boldsymbol{\chi}_1\cdot\boldsymbol{\tau}_{12}\|^2_{L^2(\Gamma_{12})}$$
$$\leq C\big(1 + \frac{(\mathcal{R}_1+\mathcal{R}_2)^2}{\mu^2}\big)\|\nabla\boldsymbol{\zeta}_1\|^2_{L^2(\Omega_1)} + C\frac{1}{\mu}|||\eta_2|||^2 + Ch^{2k_2}\|\varphi_2\|^2_{H^{k_2+1}(\Omega_2)} + C\frac{1}{\mu}\|\eta_1\|^2_{L^2(\Omega_1)}.$$

The final result is obtained by using the approximation properties (51), (53), (55), a trace theorem and the inequalities:

$$\begin{aligned}\|\boldsymbol{D}(\boldsymbol{u}_1-\boldsymbol{U}_1)\|^2_{L^2(\Omega_1)} &\le C\|\boldsymbol{D}(\boldsymbol{\chi}_1)\|^2_{L^2(\Omega_1)}+C\|\boldsymbol{D}(\boldsymbol{\zeta}_1)\|^2_{L^2(\Omega_1)},\\ \|(\boldsymbol{u}_1-\boldsymbol{U}_1)\cdot\boldsymbol{\tau}_{12}\|^2_{L^2(\Gamma_{12})} &\le C\|(\boldsymbol{\chi}_1)\cdot\boldsymbol{\tau}_{12}\|^2_{L^2(\Gamma_{12})}+C\|(\boldsymbol{\zeta}_1)\cdot\boldsymbol{\tau}_{12}\|^2_{L^2(\Gamma_{12})},\\ |||p_2-P_2|||^2 &\le C|||\xi_2|||^2+C|||\eta_2|||^2.\end{aligned}$$

□

A straight consequence of Lemma 7 and Theorem 14 is a bound on the pressure error.

Corollary 15. *Under the assumptions of Theorem 14 and if the function p_{D} belongs to $H^{k_2+1}(\Omega_2)$, there exists a constant C independent of h and μ such that*

$$\begin{aligned}|||p_2-P_2|||^2 \quad\le\quad & C\big(1+\frac{(\mathcal{R}_1+\mathcal{R}_2)^2}{\mu^2}\big)h^{2k_1}\|\boldsymbol{u}_1\|^2_{H^{k_1+1}(\Omega_1)}+C\big(1+\frac{1}{\mu}\big)h^{2k_2}\|\varphi_2\|^2_{H^{k_2+1}(\Omega_2)}\\ &+Ch^{2k_2}\|p_{\mathrm{D}}\|^2_{H^{k_2+1}(\Omega_2)}+C\frac{1}{\mu}h^{2k_1}\|p_1\|^2_{H^{k_1}(\Omega_1)}.\end{aligned}$$

Theorem 16. *Under the assumptions of Theorem 14 and Corollary 15, there exists a constant C independent of h such that*

$$\|p_1-P_1\|_{L^2(\Omega_1)}\le Ch^{k_1}\|p_1\|_{H^{k_1}(\Omega_1)}+Ch^{k_1}\|\boldsymbol{u}_1\|_{H^{k_1+1}(\Omega_1)}+Ch^{k_2}(\|\varphi_2\|_{H^{k_2+1}(\Omega_2)}+\|p_{\mathrm{D}}\|_{H^{k_2+1}(\Omega_2)}).$$

Proof. Using the same notation as in the proof of Theorem 14, we can rewrite the error equation by taking $q_2=0$:

$$\begin{aligned}b_{\mathrm{NS}}(\boldsymbol{v}_1,\xi_1) \quad=\quad & b_{\mathrm{NS}}(\boldsymbol{v}_1,\eta_1)+a_{\mathrm{NS}}(\boldsymbol{u}_1-\boldsymbol{U}_1,\boldsymbol{v}_1)-\frac{1}{2}(\boldsymbol{u}_1\cdot\boldsymbol{u}_1-\boldsymbol{U}_1\cdot\boldsymbol{U}_1,\boldsymbol{v}_1\cdot\boldsymbol{n}_{12})_{\Gamma_{12}}\\ &+c_{\mathrm{NS}}(\boldsymbol{u}_1;\boldsymbol{u}_1,\boldsymbol{v}_1)-c_{\mathrm{NS}}(\boldsymbol{U}_1;\boldsymbol{U}_1,\boldsymbol{v}_1)+(\varphi_2-\Phi_2,\boldsymbol{v}_1\cdot\boldsymbol{n}_{12})_{\Gamma_{12}}+\frac{1}{G}((\boldsymbol{u}_1-\boldsymbol{U}_1)\cdot\boldsymbol{\tau}_{12},\boldsymbol{v}_1\cdot\boldsymbol{\tau}_{12})_{\Gamma_{12}}.\end{aligned}$$

We now bound all terms in the right-hand side. Cauchy-Schwarz's inequality yields simply

$$\begin{aligned}b_{\mathrm{NS}}(\boldsymbol{v}_1,\eta_1) &\le C\|\nabla\boldsymbol{v}_1\|_{L^2(\Omega_1)}\|\eta_1\|_{L^2(\Omega_1)},\\ a_{\mathrm{NS}}(\boldsymbol{u}_1-\boldsymbol{U}_1,\boldsymbol{v}_1) &\le C\mu\|\nabla\boldsymbol{v}_1\|_{L^2(\Omega_1)}\|\boldsymbol{D}(\boldsymbol{u}_1-\boldsymbol{U}_1)\|_{L^2(\Omega_1)}.\end{aligned}$$

The nonlinear terms are handled like the term A_1 in (79).

$$\begin{aligned}\frac{1}{2}(\boldsymbol{u}_1\cdot\boldsymbol{u}_1-\boldsymbol{U}_1\cdot\boldsymbol{U}_1,\boldsymbol{v}_1\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} \quad&=\quad \frac{1}{2}(\boldsymbol{U}_1\cdot\boldsymbol{\chi}_1,\boldsymbol{v}_1\cdot\boldsymbol{n}_{12})_{\Gamma_{12}}+\frac{1}{2}(\boldsymbol{\chi}_1\cdot\boldsymbol{u}_1,\boldsymbol{v}_1\cdot\boldsymbol{n}_{12})_{\Gamma_{12}}\\ &\quad\quad -\frac{1}{2}(\boldsymbol{U}_1\cdot\boldsymbol{\zeta}_1,\boldsymbol{v}_1\cdot\boldsymbol{n}_{12})_{\Gamma_{12}}-\frac{1}{2}(\boldsymbol{\zeta}_1\cdot\boldsymbol{u}_1,\boldsymbol{v}_1\cdot\boldsymbol{n}_{12})_{\Gamma_{12}}\\ &\le\quad \frac{C(\mathcal{R}_1+\mathcal{R}_2)}{\sqrt{\mu}}\|\nabla\boldsymbol{v}_1\|_{L^2(\Omega_1)}(\|\boldsymbol{D}(\boldsymbol{\chi}_1)\|_{L^2(\Omega_1)}+\|\nabla\boldsymbol{\zeta}_1\|_{L^2(\Omega_1)}),\\ c_{\mathrm{NS}}(\boldsymbol{u}_1;\boldsymbol{u}_1,\boldsymbol{v}_1)-c_{\mathrm{NS}}(\boldsymbol{U}_1;\boldsymbol{U}_1,\boldsymbol{v}_1) \quad&=\quad c_{\mathrm{NS}}(\boldsymbol{U}_1;\boldsymbol{\chi}_1,\boldsymbol{v}_1)+c_{\mathrm{NS}}(\boldsymbol{\chi}_1;\boldsymbol{u}_1,\boldsymbol{v}_1)\\ &\quad\quad -c_{\mathrm{NS}}(\boldsymbol{U}_1;\boldsymbol{\zeta}_1,\boldsymbol{v}_1)-c_{\mathrm{NS}}(\boldsymbol{\zeta}_1;\boldsymbol{u},\boldsymbol{v}_1)\\ &\le\quad \frac{C(\mathcal{R}_1+\mathcal{R}_2)}{\sqrt{\mu}}\|\nabla\boldsymbol{v}_1\|_{L^2(\Omega_1)}(\|\boldsymbol{D}(\boldsymbol{\chi}_1)\|_{L^2(\Omega_1)}+\|\nabla\boldsymbol{\zeta}_1\|_{L^2(\Omega_1)}).\end{aligned}$$

Finally, the last two terms are bounded as:

$$\begin{aligned}(\varphi_2-\Phi_2,\boldsymbol{v}_1\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} &\le C(|||\xi_2|||+\|\eta_2\|_{L^2(\Gamma_{12})})\|\nabla\boldsymbol{v}_1\|_{L^2(\Omega_1)},\\ \frac{1}{G}((\boldsymbol{u}_1-\boldsymbol{U}_1)\cdot\boldsymbol{\tau}_{12},\boldsymbol{v}_1\cdot\boldsymbol{\tau}_{12})_{\Gamma_{12}} &\le C\|(\boldsymbol{u}_1-\boldsymbol{U}_1)\cdot\boldsymbol{\tau}_{12}\|_{L^2(\Omega_1)}\|\nabla\boldsymbol{v}_1\|_{L^2(\Omega_1)}.\end{aligned}$$

Therefore, we obtain:

$$b_{\mathrm{NS}}(\boldsymbol{v}_1,\xi_1) \le C\Theta\|\nabla \boldsymbol{v}_1\|_{L^2(\Omega_1)},$$

with

$$\Theta = \|\eta_1\|_{L^2(\Omega_1)} + \mu\|\boldsymbol{D}(\boldsymbol{u}_1-\boldsymbol{U}_1)\|_{L^2(\Omega_1)} + \frac{\mathcal{R}_1+\mathcal{R}_2}{\sqrt{\mu}}(\|\boldsymbol{D}(\boldsymbol{\chi}_1)\|_{L^2(\Omega_1)} + \|\nabla\boldsymbol{\zeta}_1\|_{L^2(\Omega_1)})$$
$$+|||\xi_2||| + \|\eta_2\|_{L^2(\Gamma_{12})} + \|(\boldsymbol{u}_1-\boldsymbol{U}_1)\cdot\boldsymbol{\tau}_{12}\|_{L^2(\Omega_1)}.$$

The inf-sup condition (46) then yields

$$\|\xi_1\|_{L^2(\Omega_1)} \le \frac{C}{\beta_*}\Theta.$$

Using the approximation results (51), (53), (54) and Theorem 14, we can conclude. □

The convergence of the solution of problem (W_B^h) is obtained in a similar fashion. The derivation of the error estimates involves less terms, for instance the term A_1 in (79) does not appear. We state only the results.

Theorem 17. *Assume that the solution to problem (W_B) is smooth enough, i.e. $\boldsymbol{u}_1 \in (H^{k_1+1}(\Omega_1))^2, p_1 \in H^{k_1}(\Omega_1)$ and $p_2 = \varphi_2 + p_{\mathrm{D}}$ with $\varphi_2 \in H^{k_2+1}(\Omega_2)$ and p_{D} belongs to $H^{k_2+1}(\Omega_2)$. Let $\mathcal{R}_1$ be defined by (41) and let $\mathcal{R}_2$ be defined by (72). Assume that the data satisfies:*

$$\mu^{3/2} > \frac{C_1^3}{\sqrt{2}}(\mathcal{P}_4^2 + \frac{1}{2}C_2C_4^2)(\mathcal{R}_1+\mathcal{R}_2).$$

Let $(\boldsymbol{U}_1,P_1,P_2)$ be a solution to problem (W_B^h). Then, there exists a constant C independent of h and μ such that

$$\mu\|\boldsymbol{D}(\boldsymbol{u}_1-\boldsymbol{U}_1)\|^2_{L^2(\Omega_1)} + |||p_2-P_2|||^2 + \|(\boldsymbol{u}_1-\boldsymbol{U}_1)\cdot\boldsymbol{\tau}_{12}\|^2_{L^2(\Gamma_{12})} \le C\big(1+\frac{(\mathcal{R}_1+\mathcal{R}_2)^2}{\mu^2}\big)h^{2k_1}\|\boldsymbol{u}_1\|^2_{H^{k_1+1}(\Omega_1)}$$
$$+C\big(1+\frac{1}{\mu}\big)h^{2k_2}\|\varphi_2\|^2_{H^{k_2+1}(\Omega_2)} + C\frac{1}{\mu}h^{2k_1}\|p_1\|^2_{H^{k_1}(\Omega_1)} + Ch^{2k_2}\|p_{\mathrm{D}}\|^2_{H^{k_2+1}(\Omega_2)}.$$

In addition, there exists a constant C independent of h such that

$$\|p_1-P_1\|_{L^2(\Omega_1)} \le Ch^{k_1}\|p_1\|_{H^{k_1}(\Omega_1)} + Ch^{k_1}\|\boldsymbol{u}_1\|_{H^{k_1+1}(\Omega_1)} + Ch^{k_2}(\|\varphi_2\|_{H^{k_2+1}(\Omega_2)} + \|p_{\mathrm{D}}\|_{H^{k_2+1}(\Omega_2)}).$$

5 Numerical examples

In the following examples, we consider the domain $(0,1)\times(0,2)$ divided into two subdomains by the interface $\Gamma_{12} = (0,1)\times\{1\}$. The Navier-Stokes region is the top part $\Omega_1 = (0,1)\times(1,2)$ whereas the Darcy region is the bottom part $\Omega_2 = (0,1)\times(0,1)$. We first verify our theoretical results by computing convergence rates for known smooth solutions. The mini elements are used in the Navier-Stokes region and the discontinuous piecewise linears are used in the Darcy region. We choose the SIPG method with a penalty parameter equal to 1 everywhere.
Solution 1: For the model W_A (with inertial forces), the exact solution is chosen as:

$$\boldsymbol{u}_1 = (y^2-2y+2x, x^2-x+2y),\quad p_1 = -x^2y+xy+y^2,\quad p_2 = 4-x^2y+xy+y^2+0.5((2x-1)^2+(x^2-x-2)^2).$$

Solution 2: For the model W_B (without inertial forces), the exact solution is similar. The pressure is slightly modified:

$$\boldsymbol{u}_1 = (y^2-2y+2x, x^2-x+2y),\quad p_1 = -x^2y+xy+y^2-4,\quad p_2 = x^2+xy+y^2$$

$1/h$	$\|\|P_2 - p_2\|\|_{L^2(\Omega_2)}$	$\|\|\boldsymbol{U}_1 - \boldsymbol{u}_1\|\|_{L^2(\Omega_1)}$	$\|\|P_1 - p_1\|\|_{L^2(\Omega_1)}$	$\|\|\boldsymbol{D}(\boldsymbol{U}_1 - \boldsymbol{u}_1)\|\|_{L^2(\Omega_1)}$	$\|\|\boldsymbol{U}_2 - \boldsymbol{u}_2\|\|_{L^2(\Omega_2)}$
2	1.054×10^{-1}	6.426×10^{-2}	3.271×10^{-1}	2.793×10^{-1}	3.547×10^{-1}
4	2.592×10^{-2}	1.598×10^{-2}	6.640×10^{-2}	1.367×10^{-1}	1.643×10^{-1}
8	6.417×10^{-3}	4.001×10^{-3}	2.047×10^{-2}	6.794×10^{-2}	7.911×10^{-2}
16	1.594×10^{-3}	1.000×10^{-3}	6.658×10^{-3}	3.390×10^{-2}	3.896×10^{-2}
32	3.970×10^{-4}	2.499×10^{-4}	2.304×10^{-3}	1.594×10^{-2}	1.961×10^{-2}
rate	2.00	2.00	1.53	1.00	1.00

Table 1: Numerical errors and convergence rates for solution 1 and the choice $k_2 = 1$.

$1/h$	$\|\|P_2 - p_2\|\|_{L^2(\Omega_2)}$	$\|\|\boldsymbol{U}_1 - \boldsymbol{u}_1\|\|_{L^2(\Omega_1)}$	$\|\|P_1 - p_1\|\|_{L^2(\Omega_1)}$	$\|\|\boldsymbol{D}(\boldsymbol{U}_1 - \boldsymbol{u}_1)\|\|_{L^2(\Omega_1)}$	$\|\|\boldsymbol{U}_2 - \boldsymbol{u}_2\|\|_{L^2(\Omega_2)}$
2	1.052×10^{-1}	6.426×10^{-2}	3.345×10^{-1}	2.793×10^{-1}	3.507×10^{-1}
4	2.598×10^{-2}	1.598×10^{-2}	6.889×10^{-2}	1.366×10^{-1}	1.625×10^{-1}
8	6.418×10^{-3}	3.998×10^{-3}	2.909×10^{-2}	6.794×10^{-2}	7.811×10^{-2}
16	1.592×10^{-3}	9.989×10^{-4}	6.754×10^{-3}	3.390×10^{-2}	3.841×10^{-2}
32	3.963×10^{-4}	2.495×10^{-4}	2.248×10^{-3}	1.694×10^{-2}	1.907×10^{-2}
rate	2.00	2.00	1.58	1.00	1.00

Table 2: Numerical errors and convergence rates for solution 2 and the choice $k_2 = 1$.

Table 1 and Table 2 give the numerical rates for both models. We verify that our numerical rates correspond to the theoretical results, namely $\mathcal{O}(h)$ for the Navier-Stokes velocity error in the gradient norm and for the Darcy pressure error in the gradient norm. One of the benefits of using discontinuous Galerkin is that one can easily increase the polynomial degree. We repeat the same experiments above and increase the polynomial degree in the Darcy region to two. We choose the solution 1 and present the results in Table 3.We observe that the solution is more accurate in the Darcy region as the polynomial degree increases. Furthermore, the errors in the Darcy region locally converge faster than the errors in the Navier-Stokes region. We obtain a rate of $\mathcal{O}(h^2)$ for the pressure error in the gradient norm. Our global error estimates guarantee only the rate $\mathcal{O}(h)$. We have observed similar results for the solution 2.

In the next example, we consider the following Dirichlet boundary conditions for the Navier-Stokes region:

$$\boldsymbol{u}_1 = (\sin(\pi x), 0), \quad \text{on} \quad (0,1) \times \{2\},$$
$$\boldsymbol{u}_1 = (0,0), \quad \text{on} \quad (\{0\} \times (1,2)) \cup (\{1\} \times (1,2))$$

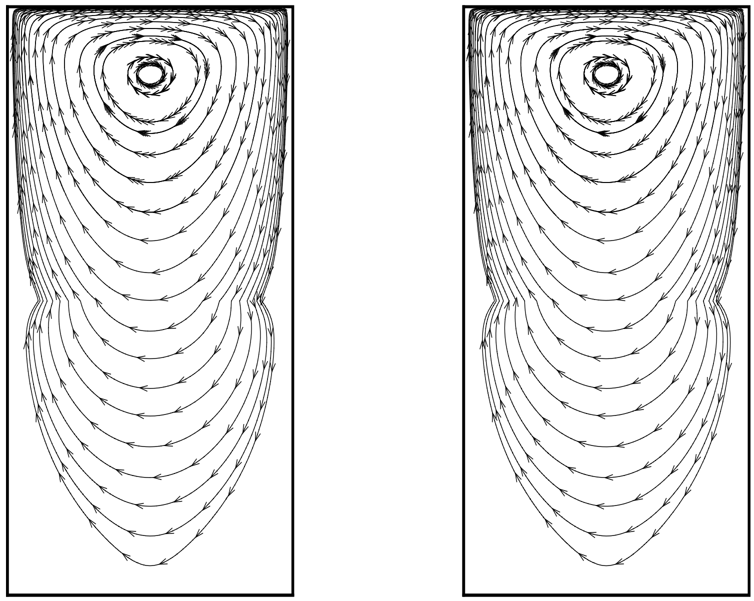
For the Darcy region, we assume zero Neumann boundary condition for the vertical boundaries and zero Dirichlet boundary condition for the horizontal boundary. We use the mini elements for the Navier-Stokes region and discontinuous polynomials of degree one for the Darcy region. The viscosity is chosen to be equal to one. The mesh consists of a uniform triangularization of the domain with 8192 triangles. In Fig. 1, we show the velocity streamlines obtained from the schemes W_A^h and W_B^h. They are almost identical to each other. For a better comparison, we compute the difference between the two solutions. Fig. 2(a) shows the contours of the difference between the two approximations of the x-component of the velocity field. We observe that

$1/h$	$\|\|P_2 - p_2\|\|_{L^2(\Omega_2)}$	$\|\|\boldsymbol{U}_1 - \boldsymbol{u}_1\|\|_{L^2(\Omega_1)}$	$\|\|P_1 - p_1\|\|_{L^2(\Omega_1)}$	$\|\|\boldsymbol{D}(\boldsymbol{U}_1 - \boldsymbol{u}_1)\|\|_{L^2(\Omega_1)}$	$\|\|\boldsymbol{U}_2 - \boldsymbol{u}_2\|\|_{L^2(\Omega_2)}$
2	2.667×10^{-2}	6.425×10^{-2}	3.343×10^{-1}	2.793×10^{-1}	5.982×10^{-2}
4	7.276×10^{-3}	1.600×10^{-2}	6.766×10^{-2}	1.366×10^{-1}	1.584×10^{-2}
8	1.872×10^{-3}	4.004×10^{-3}	2.072×10^{-2}	6.794×10^{-2}	4.049×10^{-3}
16	4.726×10^{-4}	1.000×10^{-3}	6.702×10^{-3}	3.390×10^{-2}	1.022×10^{-3}
32	1.187×10^{-4}	2.499×10^{-4}	2.230×10^{-3}	1.694×10^{-2}	2.570×10^{-4}
rate	2.00	2.00	1.58	1.00	2.00

Table 3: Numerical errors and convergence rates for solution 1 and the choice $k_2 = 2$.

the difference is very small, of the order 10^{-5}. A similar comment can be made about the difference between the two approximations of the y-component of the velocity field (see Fig. 2(b)). The difference between the two approximations of the pressure is slightly larger, namely of the order 10^{-3}.

Finally we repeat the same experiment but we set the fluid viscosity equal to 0.005. Fig. 4 shows the difference between the approximations of the two models. Overall the difference is small, of the order 10^{-5}. At some localized areas near the interface, the difference increases to 10^{-3}.



(a) (b)

Figure 1: Streamlines for the numerical velocity for the model without inertial forces (a) and with inertial forces for viscosity equal to 1.

6 Conclusions

We define and analyze two model problems for the coupled system of Navier-Stokes and Darcy equations. We formulate a method that combines the classical conforming finite element method for Navier-Stokes with the discontinuous Galerkin method for Darcy. If inertial forces are included in the balance of forces across the interface, existence of weak and numerical solutions is obtained unconditionally. Small data condition is needed if one does not take into account inertial forces and in addition uniqueness is obtained locally. Convergence of the discrete solution is proved with respect to the mesh size. The meshes on the interface can be non-matching. This is an attractive feature if one implements the method using a domain decomposition approach. From a numerical point of view, the two schemes yield similar solutions for the velocity with a difference of very small value in the sup norm.

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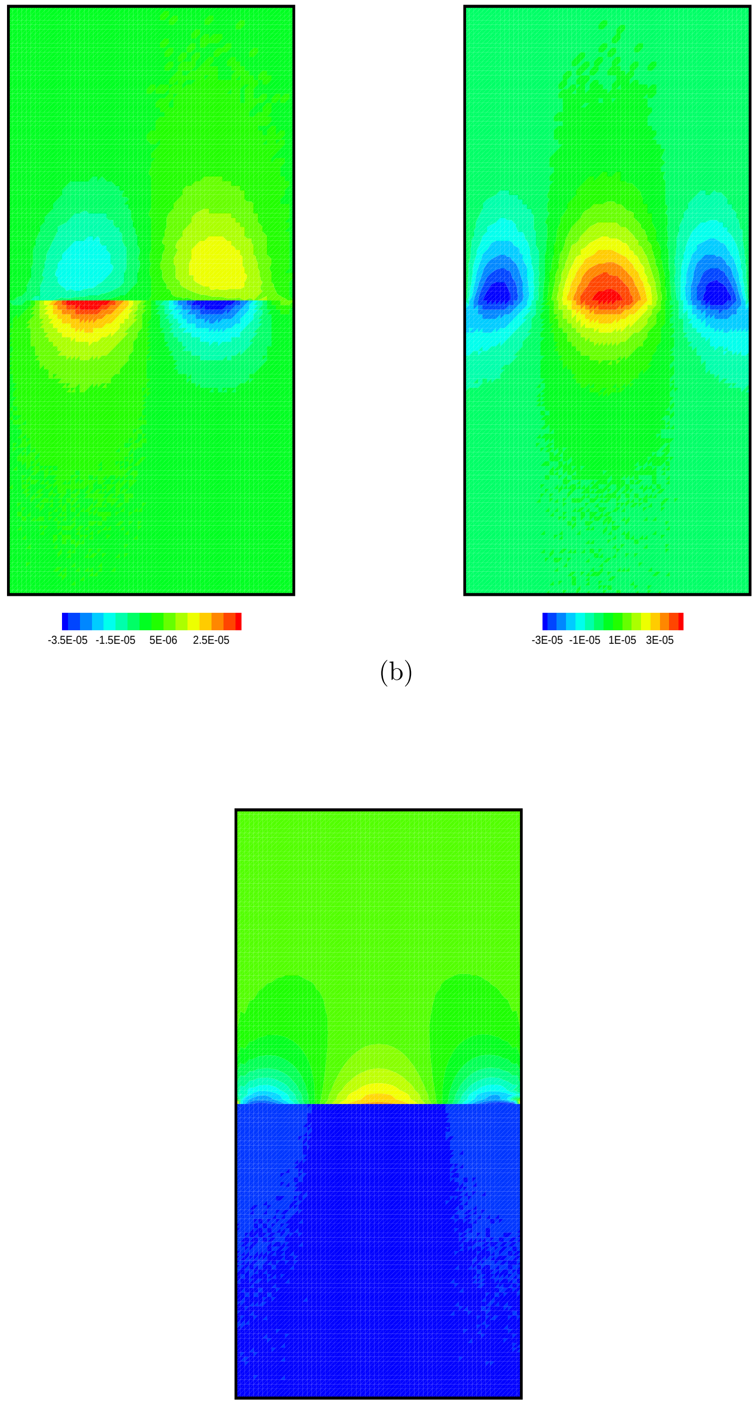
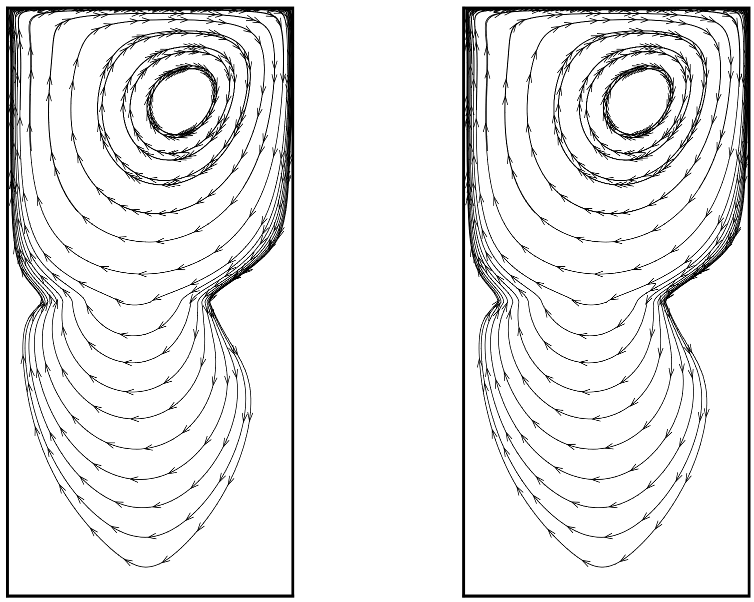


(a) (b) (c)

Figure 2: Difference between the solutions obtained from the two models for viscosity equal to 1: (a) x-component of velocity, (b) y-component of velocity and (c) pressure.



(a) (b)

Figure 3: Streamlines for the numerical velocity for the model without inertial forces (a) and with inertial forces for viscosity equal to 0.005.

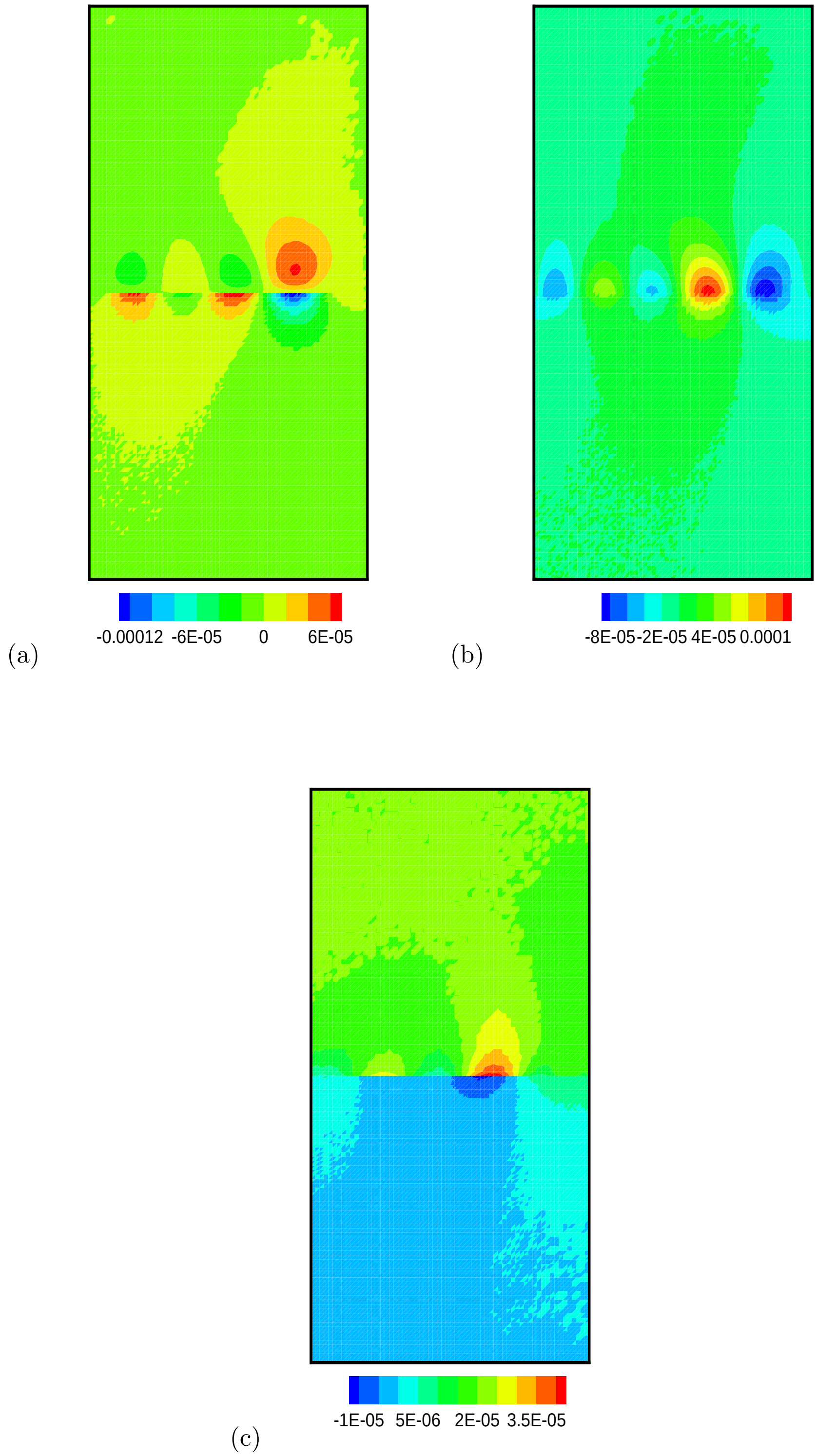


Figure 4: Difference between the solutions obtained from the two models for viscosity equal to 0.005: (a) x-component of velocity, (b) y-component of velocity and (c) pressure.