Integral Values of Generating Functions Related to the Fibonacci Sequence

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Introduction

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In a recent article in the College Mathematics Journal, Prapanpong Pongsriiam [**4**] proved a conjecture of Dae Hong [**2**] about integral values of the generating functions of the Fibonacci and Lucas numbers. Specifically, for both generating functions, he found all rational values of x such that the function evaluated at x will be an integer. In this article, we generalize his methods and prove similar results for a family of related sequences. Recall that the standard definitions for the Fibonacci sequence F_i and the Lucas sequence L_i are

$$\begin{array}{llll} F_0 = 0, & F_1 = 1, & \text{and} & F_i = F_{i-1} + F_{i-2} \text{ for } i \geq 2 \\ L_0 = 2, & L_1 = 1, & \text{and} & L_i = L_{i-1} + L_{i-2} \text{ for } i \geq 2. \end{array}$$

For a positive integer $a \geq 3$, we define the Fibonacci-like sequence G_i^a by

$$G_0^a = a, \quad G_1^a = 1, \quad \text{and} \quad G_i^a = G_{i-1}^a + G_{i-2}^a \text{ for } i \geq 2.$$

For example, the G_i^3-sequence begins

$$3, 1, 4, 5, 9, 14, 23, 37, 60, 97, 157, 254, 411, \ldots.$$

Note that we can define any of these sequences for negative indices by using the recurrence relation. For example, the relation says that we should have $G_1^3 = G_0^3 + G_{-1}^3$, and so we obtain $G_{-1}^3 = -2$.

For each of these sequences, our goal is to find all of the rational x-values which make its generating function an integer. We will show how to find all of these x-values and find several families of solutions which generalize the ones given by Pongsriiam and Hong. For many values of the parameter a, these are the only solutions, but many values of a yield other solutions as well. Any additional solutions also come in families, and we will see how to find these families, but we do not have a simple formula which tells us their elements. In particular, we will prove the following theorems.

Theorem 1. *Let $a \geq 3$ be a positive integer, and define the sequence G_i^a as above. Let $G^a(x)$ be the generating function for this sequence. Then the values $x = F_i/F_{i+1}$ produce integer values of $G^a(x)$ for all integers $i \neq 0$, and the values $x = -G_{i+1}^a/G_i^a$ produce integer values of $G^a(x)$ for all integers i (including $i = 0$). Depending on the value of a, there may be other families of solutions as well. These solutions can be obtained from solutions of the Diophantine equation $m^2 + 3mc + c^2 - (a^2 + a - 1) = 0$ with $2 \leq m \leq a - 2$.*

`In the LaTeX code, this theorem is labeled ‘‘thm.’’` TODO

Theorem 2. *Suppose that the number a and the sequence G_i^a are defined as in Theorem 1, and that $a \equiv 2 \pmod 5$. Let L_i be the Lucas sequence defined above. Furthermore, write $a = 5n + 2$ and define the sequence H_i by*

$$H_0 = n, \quad H_1 = 3n + 1, \quad \text{and} \quad H_i = H_{i-1} + H_{i-2} \text{ for all } i.$$

Then the values $x = L_i/L_{i+1}$ and $x = H_i/H_{i+1}$ produce integer values of $G^a(x)$ for all integers i.

`In the LaTeX code, this theorem is labeled ‘‘thm2.’’` TODO

The remainder of this article is divided into 4 sections. Since it is easy to get bogged down in the details of the proof, in the next section we will explain the ideas behind the proof for the specific value $a = 4$. Then we'll list the properties of Fibonacci and Lucas numbers that we'll need for the proofs of our results. After that, our main section will give the proofs of the theorems. The short final section contains some avenues for future work, some of which may be suitable for undergraduate research projects.

The case $a = 4$

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In this section, we'll show how to solve our problem in the specific case when $a = 4$. Our goal here is to focus on ideas, not to give the details of our proofs. In a lot of places, we will simply state results without proving them. We will give complete proofs of more general statements later.

When $a = 4$, we are interested in the sequence G_i^4 with $G_0^4 = 4$, $G_1^4 = 1$, and $G_i^4 = G_{i-1}^4 + G_{i-2}^4$ when $i \geq 2$. This sequence begins

$$4, 1, 5, 6, 11, 17, 28, 45, 73, 118, 191, 309, \ldots.$$

As mentioned in the introduction, we can also use the recurrence relation to define G_i^4 for negative values of i, by setting $G_i^4 = G_{i+2}^4 - G_{i+1}^4$ when i is negative. This gives us the values

$$G_{-1}^4 = -3, \quad G_{-2}^4 = 7, \quad G_{-3}^4 = -10, \quad G_{-4}^4 = 17, \quad G_{-5}^4 = -27, \ldots.$$

The generating function for this sequence is

$$G^4(x) = \frac{-3x + 4}{1 - x - x^2}.$$

Following Pongsriiam, suppose that k is an integer and that we want to solve the equation $G^4(x) = k$. Cross-multiplying and collecting like terms leads to the quadratic equation

$$kx^2 + (k-3)x + (4-k) = 0, \tag{1}$$

which has solutions

$$x = \frac{(3-k) \pm \sqrt{5k^2 - 22k + 9}}{2k}.$$

`In the LaTeX code, this equation is labeled ‘‘quad-a=4.’’` TODO

We can see that x is rational if and only if $5k^2 - 22k + 9$ is a perfect square, and so we need to discover which values of k make this happen. Suppose that m is a positive number so that $5k^2 - 22k + 9 = m^2$. Then we have $25k^2 - 110k + 45 = 5m^2$, and completing the square on the left-hand side leads to

$$(5k-11)^2 = 5m^2 + 76.$$

Let's study the values of m such that $5m^2 + 76$ is a perfect square. With a little bit of work, we can show that this happens if and only if there is an integer n such that

$$m^2 - 3mn + n^2 = 19. \tag{2}$$

`In the LaTeX code, this equation is labeled ‘‘mn-for-a=4.’’` TODO

Hence we need to study the solutions of this new equation (2).

If (m, n) is a solution of (2), then so is $(n, 3n - m)$, and hence we can use the transformation $(m, n) \mapsto (n, 3n - m)$ to produce families of increasing positive values of m such that $5m^2 + 76$ is a square. In order to use the transformation to produce all of the possible values of m, we need to find all of the initial pairs (m, n) for these families. It turns out that (m, n) is the smallest pair in a family if and only if we have $n = 3m + c$ with $c > 0$. Making this substitution in (2) leads to the equation

$$m^2 + 3mc + c^2 = 19. \tag{3}$$

Since (3) has all coefficients positive, and both m and c need to be positive integers, we can solve this equation completely through simple trial and error! The only two solutions are $(m, c) = (1, 3)$ and $(m, c) = (3, 1)$. These lead to the following families of solutions of (2):

- 1, 6, 17, 45, 118, 309, 809, 2118, 5545, 14517,...
- 3, 10, 27, 71, 186, 487, 1275, 3338, 8739, 22879....

Since we have found all of the solutions of (3), we know that we have all the positive integer solutions of (2), and hence these values are all the positive integers m such that $5m^2 + 76$ is a perfect square. In either sequence, if we label the terms as $M_0, M_1, \ldots$, we can see that the terms satisfy the recurrence relation

$$M_i = 3M_{i-1} - M_{i-2}. \tag{4}$$

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‘‘recurrence-a=4.’’
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TODO

When we prove our theorems in general, we will show that there is always a family of solutions starting with 1 and another family of solutions starting with $a-1$. While these are the only families for $a=4$, there may be more families for other values of a. However, no matter how many families there are, the members of each family satisfy this same recurrence relation.

These families give all the solutions to the equation (2), but unfortunately not every solution of (2) yields a rational value of x for our equation $G(x)=k$. This is because, writing $S_i=\sqrt{5M_i^2+76}$, we need to have either $5k-11=S_i$ or $5k-11=-S_i$, and $11 \pm S_i$ might not be a multiple of 5. Defining $K_i^+=(11+S_i)/5$ and $K_i^-=(11-S_i)/5$, we seek the subscripts i such that K_i^+ and K_i^- are integers. For each family, we can make a table of the values of i, M_i, K_i^+, and K_i^- to gain some intuition. These tables are found below. We can see that if $M_0=1$, then K_i^+ is an integer exactly when i is even and K_i^- is an integer exactly when i is odd. If $M_0=3$ then K_i^+ is an integer exactly when i is odd, and K_i^- is an integer exactly when i is even. We will show later that this pattern continues for all i.

i	M_i	$S_i=\sqrt{5M_i^2+76}$	K_i^+	K_i^-
0	1	9	4	0.4
1	6	16	5.4	-1
2	17	39	10	-5.6
3	45	101	22.4	-18
4	118	264	55	-50.6
5	309	691	140.4	-136
6	809	1809	364	-359.6
7	2118	4736	949.4	-945
8	5545	12399	2482	-2477.6
9	14517	32461	6494.4	-6490

TABLE 1: **Values of K_i^+ and K_i^- in the $M_0=1$ family**

We note in passing that in each table, the values of $S_i=\sqrt{5M_i^2+76}$ are given by the formula $S_i=F_{2i}S_1-F_{2i-2}S_0$ and satisfy the recursion $S_i=3S_{i-1}-S_{i-2}$. We don't need this fact here, but it will be important later when we prove which values of K_i^+ and K_i^- are integral. As another aside, when we prove our theorems we will see that if $a\equiv 2 \pmod 5$, then both K_i^+ and K_i^- are integers for all i. This explains the additional families of solutions for these values of a in Theorem 2.

Now that we have found all the values of M_i and k which lead to rational values of x, we need to actually find these x-values. For each i, define K_i (with no superscript) to be either K_i^+ or K_i^-, whichever is an integer. Then the numbers K_i are exactly the values of k for which the equation $G^4(x)=k$ has rational solutions, and these

i	M_i	$S_i = \sqrt{5M_i^2 + 76}$	K_i^+	K_i^-
0	3	11	4.4	0
1	10	24	7	-2.6
2	27	61	14.4	-10
3	71	159	34	-29.6
4	186	416	85.4	-81
5	487	1089	220	-215.6
6	1275	2851	572.4	-568
7	3338	7464	1495	-1490.6
8	8739	19541	3910.4	-3906
9	22879	51159	10234	-10229.6

TABLE 2: **Values of K_i^+ and K_i^- in the $M_0 = 3$ family**

solutions are given by

$$x = \frac{3 - K_i \pm M_i}{2K_i}.$$

We can calculate the values of x that this formula yields, and these values are in the tables below.

i	M_i	K_i	x using $+M_i$	x using $-M_i$
0	1	4	0/1	-1/4
1	6	-1	-5/1	1/1
2	17	10	1/2	-6/5
3	45	-18	-11/6	2/3
4	118	55	3/5	-17/11
5	309	-136	-28/17	5/8
6	809	364	8/13	-45/28
7	2118	-945	-73/45	13/21
8	5545	2482	21/34	-118/73
9	14517	-6490	-191/118	34/55

TABLE 3: **Values of x in the $M_0 = 1$ family**

Note that when $k = K_i = 0$, our method gives undefined values for x. This is expected, since the quadratic formula assumes that the x^2 term has a nonzero coefficient, which is not the case here. However, it is easy to see that the only solution of $G^4(x) = 0$ is $x = 4/3$. We can now guess formulas for the values of x. When $M_0 = 1$, each row has a column in which the value is F_i/F_{i+1}. When $M_0 = 3$, each row except $i = 0$ has a column in which the value is $-F_{i+1}/F_i$. However, by the identity (6) in the next section, we have $-F_{i+1}/F_i = F_{-(i+1)}/F_{-i}$. Together, these values yield F_i/F_{i+1} for all integers i except $i = 0$. Similarly, the other values in the table are $-G^4_{i+1}/G^4_i$ for all $i \neq -1$, while $-G^4_0/G^4_{-1} = 4/3$ is the value above corresponding to $k = 0$. These are exactly the x-values claimed in Theorem 1.

i	M_i	K_i	x using $+M_i$	x using $-M_i$
0	3	0		
1	10	7	3/7	-1/1
2	27	-10	-2/1	7/10
3	71	34	10/17	-3/2
4	186	-81	-5/3	17/27
5	487	220	27/44	-8/5
6	1275	-568	-13/8	44/71
7	3338	1495	71/115	-21/13
8	8739	-3906	-34/21	115/186
9	22879	10234	186/301	-55/34

TABLE 4: **Values of x in the $M_0 = 3$ family**

Preliminaries for the proof

`In the LaTeX code, this section is labeled ‘‘sec-prelim.’’` TODO

In this section, we'll give the identities for Fibonacci and Lucas numbers that we'll need to prove our theorems. All the results that we'll need for the proof, except apparently (13) and (14), either can be found (perhaps in a slightly different form) in the excellent reference [**3**], or else can be trivially derived from identities found in [**3**]. The identity (14) follows immediately from the identity (13), which can be easily proved by induction on n.

Let F_i and L_i be the Fibonacci and Lucas sequences defined in the introduction, and for a fixed integer $a \geq 3$, define the sequence G_i^a by

$$G_0^a = a, \quad G_1^a = 1, \quad \text{and} \quad G_i^a = G_{i-1}^a + G_{i-2}^a \text{ for } i \geq 2.$$

We have the identity

$$G_n^a = F_{n-1}a + F_n. \tag{5}$$

As mentioned in the introduction, we can extend any of these sequences to negative indices by using the recurrence relation. We have the identities

$$F_{-n} = (-1)^{n+1}F_n \tag{6}$$

$$L_{-n} = (-1)^n L_n \tag{7}$$

$$G_{-n}^a = (-1)^n(F_{n+1}a - F_n). \tag{8}$$

For convenience, we'll almost always drop the superscript a from our notation, as there will be no chance of this causing confusion.

During our proof, we'll need the following identities for the Fibonacci and Lucas numbers:

$$F_n^2 + (-1)^n = F_{n-1}F_{n+1} \tag{9}$$

$$F_n^2 + 1 = F_{n+1}^2 - F_{n+1}F_n, \quad n \text{ even} \tag{10}$$

$$F_n^2 - 1 = F_nF_{n-1} + F_{n-1}^2, \quad n \text{ odd} \tag{11}$$

$$F_{2n+2} = 3F_{2n} - F_{2n-2} \tag{12}$$

$$F_n^2 + F_{n-2}^2 - (-1)^n = 3F_nF_{n-2} \tag{13}$$

$$F_{2n}^2 + F_{2n-2}^2 - 1 = 3F_{2n}F_{2n-2} \tag{14}$$

$$F_{2n} = F_nL_n \tag{15}$$

$$L_n = F_{n+1} + F_{n-1} \tag{16}$$

$$L_n = F_n + 2F_{n-1} \tag{17}$$

$$L_{n-1} = 2F_n - F_{n-1} \tag{18}$$

$$L_{n+1} = 3F_n + F_{n-1} \tag{19}$$

$$F_{2n+1} = F_{n+1}^2 + F_n^2. \tag{20}$$

The reader may be interested in the article [**1**], which develops an identity analogous to (14) involving odd subscripted Fibonacci numbers and shows some extremely interesting applications of it.

The proof in general

`In the LaTeX code, this section is labeled ‘‘sec-general.’’` ToDo

In this section, we will give the proofs of Theorems 1 and 2. We will use the same method as when we found the solutions when $a = 4$, but we will give complete proofs here.

First steps We begin by deriving the formula for the generating function for $G(x)$. (As noted above, we will drop the superscript from our notation unless it is needed to ensure clarity.)

Lemma 3. *The generating function for the sequence G_i is given by the formula*

$$G(x) = \frac{(1-a)x + a}{1 - x - x^2}. \tag{21}$$

`In the LaTeX code, this equation is labeled ‘‘genfn.’’` ToDo

Proof. The generating function for our sequence, written in power series form, is

$$G(x) = G_0 + G_1x + G_2x^2 + G_3x^3 + G_4x^4 + \cdots .$$

From this, we immediately have

$$xG(x) = G_0x + G_1x^2 + G_2x^3 + G_3x^4 + \cdots$$

and

$$x^2G(x) = G_0x^2 + G_1x^3 + G_2x^4 + \cdots .$$

We can then see that

$$\begin{aligned}G(x) - xG(x) - x^2G(x)\\ &= G_0 + (G_1 - G_0)x + (G_2 - G_1 - G_0)x^2 + (G_3 - G_2 - G_1)x^3 + \cdots\\ &= a + (1-a)x.\end{aligned}$$

Note that the recursion relation for the sequence ensures that all of the coefficients of the x^2 and higher degree terms are zero. The formula for $G(x)$ now follows by simple algebra. □

Now suppose that k is an integer and that we wish to solve the equation $G(x) = k$. This is equivalent to solving the quadratic equation

$$kx^2 + (1 - a + k)x + (a - k) = 0. \tag{22}$$

`In the LaTeX code, this equation is labeled ‘‘quad.’’` TODO

When $k = 0$, this equation has the unique solution $x = -a/(1-a) = -G_0/G_{-1}$. When $k \neq 0$, the solutions to (22) are given by

$$x = \frac{-(1-a+k) \pm \sqrt{5k^2 + (2-6a)k + (1-a)^2}}{2k}. \tag{23}$$

`In the LaTeX code, this equation is labeled ‘‘sln.’’` TODO

Clearly, x will be rational if and only if $5k^2 + (2-6a)k + (1-a)^2$ is a perfect square. Suppose that m is a positive integer* such that

$$5k^2 + (2-6a)k + (1-a)^2 = m^2.$$

After multiplying the above equation by 5 and completing the square on the left-hand side, we obtain

$$(5k + (1-3a))^2 = 5m^2 - 5(1-a)^2 - (1-3a)^2,$$

i.e.,

$$(5k + (1-3a))^2 = 5m^2 + 4(a^2 + a - 1). \tag{24}$$

`In the LaTeX code, this equation is labeled ‘‘5m2.’’` TODO

*Note that $m = 0$ is impossible. Having $m = 0$ would imply that $a^2 + a - 1$ is a perfect square, but this number is strictly between a^2 and $(a+1)^2$.

Finding acceptable values of m Let us now see how to find all values of m such that $5m^2 + 4(a^2 + a - 1)$ is a perfect square. For the moment, we will ignore the question of whether a given value of m leads to an integral value of k.

Lemma 4. *Let a be a fixed positive integer. Then $5m^2 + 4(a^2 + a - 1)$ is a perfect square if and only if there is an integer n such that*

$$m^2 - 3mn + n^2 - (a^2 + a - 1) = 0. \tag{25}$$

`In the LaTeX code, this lemma is labeled ‘‘equivalence.’’` TODO

`In the LaTeX code, the above equation is labeled ‘‘mn-eqn.’’` TODO

Proof. Suppose first that such an integer n exists. Considering (25) as a quadratic equation with n as a variable, we see that

$$n = \frac{3m \pm \sqrt{5m^2 + 4(a^2 + a - 1)}}{2}. \tag{26}$$

`In the LaTeX code, this equation is labeled ‘‘n.’’` TODO

Since m and n are integers, we see that $5m^2 + 4(a^2 + a - 1)$ must be a perfect square. Conversely, suppose that $5m^2 + 4(a^2 + a - 1)$ is a perfect square. We need to show that the expression for n given in (26) yields an integer. To see this, we have

$$\sqrt{5m^2 + 4(a^2 + a - 1)} \equiv 5m^2 + 4(a^2 + a - 1) \equiv m \pmod 2.$$

It immediately follows that

$$3m \pm \sqrt{5m^2 + 4(a^2 + a - 1)} \equiv 3m \pm m \equiv 0 \pmod 2,$$

and we are finished. □

Armed with this equivalence, we now study the solutions of (25). Since the equation is symmetric in the variables, it suffices to find the solutions with $m \leq n$. If $m = n$, then the left-hand side of (25) is negative, and hence (m, n) is not a solution. Therefore we can look for solutions with $m < n$. As in the $a = 4$ case, we wish to separate the solutions into families. We can do this by using the following lemma.

Lemma 5. *Suppose that (m, n) is an ordered pair of integers which satisfies* (25). *Then both $(n, 3n - m)$ and $(3m - n, m)$ are also ordered pairs of integers satisfying* (25). *Moreover, the transformations $(m, n) \mapsto (n, 3n - m)$ and $(m, n) \mapsto (3m - n, m)$ are inverses.*

`In the LaTeX code, this equation is labeled ‘‘family.’’` TODO

We won't prove this lemma since the proof just involves simple algebra.

Now, suppose that (m,n) is an integral solution of (25) with $m < n$. We wish to use the transformation

$$(m,n) \mapsto (3m-n, m) \tag{27}$$

`In the LaTeX code, this equation is labeled ‘‘trans.’’` TODO

to produce pairs with smaller and smaller (but positive) first coordinates. Once we reach an ordered pair with the smallest positive first coordinate, applying the transformation $(m,n) \mapsto (n, 3n-m)$ will give us an infinite family of solutions to (25), and thus a family of m-values with $5m^2 + 4(a^2+a-1)$ a perfect square.

There are two possible ways in which our repeated use of the transformation (27) can end. We are finished when either $3m - n < 0$, so that the new ordered pair has a negative coordinate, or when $3m - n > m$, so that our new ordered pair has $m > n$ instead of $m < n$. However, we can now prove the following lemma.

Lemma 6. *The case $3m - n > m$ cannot occur.*

`In the LaTeX code, this lemma is labeled ‘‘impossible.’’` TODO

Proof. Suppose that (m,n) is an integral solution of (25) with $m < n$ and $3m - n > m$. Then we have $m < n < 2m$, and we can write $n = m + c$, where $0 < c < m$. Plugging this into (25) gives us

$$\begin{aligned} m^2 - 3mn + n^2 - (a^2+a-1) &= m^2 - 3m(m+c) + (m+c)^2 - (a^2+a-1) \\ &= c^2 - m^2 - mc - (a^2+a-1). \end{aligned}$$

However, we have $c^2 - m^2 < 0$ since $0 < c < m$, and so our final expression is strictly negative. Hence (m,n) cannot satisfy (25), a contradiction. □

Thus our repeated use of (27) will only end when we reach an ordered pair (m,n) with $3m - n < 0$, i.e., with $n > 3m$. Suppose that (m,n) is such a pair, and write $n = 3m + c$ with $c > 0$. In this case, the equation (25) becomes

$$m^2 + 3mc + c^2 - (a^2+a-1) = 0. \tag{28}$$

`In the LaTeX code, this equation is labeled ‘‘mc-eqn.’’` TODO

Therefore, the initial values for each family of solutions to (25) correspond exactly to the positive integer solutions of (28). In the following lemma, we find two solutions of (28) which work for any value of a, and also give a best possible bound which shows us where to search for any possible additional solutions.

Lemma 7. *Two positive integral solutions of* (28) *are* $(m, c) = (1, a-1)$ *and* $(m, c) = (a-1, 1)$. *These are the only solutions with either variable equal to 1 or* $a-1$. *There are no positive integral solutions with either* $m \geq a$ *or* $c \geq a$.

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In the LaTeX code, this lemma is labeled ‘‘mc-slns.’’
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TODO

Proof. If $m = 1$, then (28) becomes $c^2 + 3c - (a^2 + a - 2) = 0$, and the left-hand side factors as $(c - (a-1))(c - (-a-2))$. The solution $c = -a - 2$ is negative, and we may discard it. Similarly, if $m = a - 1$, then (28) becomes $c^2 + (3a-3)c + (2 - 3a) = 0$, and the left-hand side factors as $(c-1)(c - (2-3a))$. Again, the solution $c = 2 - 3a$ may be discarded since it is negative. Since (28) is symmetric in m and c, the first two statements of the lemma are true. Finally, suppose that $m \geq a$. Since $c \geq 1$, we have

$$\begin{aligned} m^2 + 3mc + c^2 - (a^2 + a - 1) &\geq a^2 + 3a + 1 - (a^2 + a - 1) \\ &= 2a + 2 \\ &> 0, \end{aligned}$$

and so (m, c) cannot be a solution of (28). Symmetry again shows that there are no solutions with $c \geq a$. □

The solution $(m, c) = (1, a-1)$ of (28) leads to the solution $(m, n) = (1, a+2)$ of (25), and therefore also to a family of m-values such that $5m^2 + 4(a^2 + a - 1)$ is a perfect square. Similarly, the solution $(m, c) = (a-1, 1)$ of (28) leads to the solution $(m, n) = (a-1, 3a-2)$ of (25), and to a second family of m-values. To find other families of solutions, we can try each possible value of m with $2 \leq m \leq a - 2$ and see whether it leads to a positive integral value of c.

Computationally, it seems difficult to tell, given the value of a, whether there will be more solutions than just the two "standard" families. For $0 \leq a \leq 100$, there are 36 values with "extra" families, the smallest one being $a = 14$. For $101 \leq a \leq 200$, there are 51 values of a which have extra families of solutions. The values of a with $3 \leq a \leq 100$ which have *no* extra families are given in the table below. For these values of a, we will see that the x-values given in Theorems 1 and 2 are the only rational x such that $G^a(x)$ is an integer. Among the values which do have extra solutions, it seems most common for there to be exactly two additional families, although it is possible to have more. Up to $a = 202$, the value $a = 42$ is the only one which has only one extra family. (This arises when (28) has a single extra solution with $m = c$.)

Properties of the $\mathbf{M_i}$ sequences Our ultimate goal is to show that the m-values in our standard families lead to the values of x in the statements of the theorems. In order to do this, we need some information about the behavior of these families. Although we are only interested in the standard families, the lemmas in this section apply to any family of solutions of (25).

Let us refer to any family of m-values that can be found using the methods above as M_i, $i \geq 0$. (As with our lack of superscripts, this abuse of notation will not cause any confusion.) Our first standard family has

$$M_0 = 1 \quad \text{and} \quad M_1 = a + 2$$

3	11	21	32	46	59	72	89
4	12	22	35	48	60	76	93
5	13	24	37	50	64	77	94
6	15	26	38	53	65	82	96
7	16	27	39	54	66	83	97
8	17	28	41	55	67	85	100
9	19	30	44	56	68	86	
10	20	31	45	57	70	87	

TABLE 5: Values with $3 \le a \le 100$ and no extra solutions

and the second standard family has

$$M_0 = a - 1 \quad \text{and} \quad M_1 = 3a - 2.$$

Any of our families (standard or otherwise) satisfies the recurrence relation

$$M_i = 3M_{i-1} - M_{i-2},$$

as can be seen from the transformation $(m, n) \mapsto (n, 3n - m)$ used to produce the family. Using this recurrence relation, we can find a formula for the elements of each family. An easy induction argument using (12) allows us to prove the following lemma. We will omit the details of the proof.

Lemma 8. *For each family M_i, we have*

$$M_i = F_{2i}M_1 - F_{2i-2}M_0 \quad \text{for } i \ge 0,$$

where F_i is the Fibonacci sequence. If our family comes from the solution (M_0, c) of (28)*, then we also have*

$$M_i = F_{2i+2}M_0 + F_{2i}c \quad \text{for } i \ge 0.$$

`In the LaTeX code, this lemma is labeled ‘‘M-values.’’` TODO

In order to calculate the values of k associated with a family M_i, we will need information about the numbers $S_i = \sqrt{5M_i^2 + 4(a^2 + a - 1)}$. Here we abuse notation in the same way as with M_i, allowing S_i to refer to any family of M_i-values as needed. We show that the numbers S_i have essentially the same form as the M_i.

Lemma 9. *Let M_i be a family of positive solutions of* (25)*, as defined above. If we define $S_i = \sqrt{5M_i^2 + 4(a^2 + a - 1)}$, then we have*

$$S_i = F_{2i}S_1 - F_{2i-2}S_0 \quad \text{for } i \ge 0. \tag{29}$$

`In the LaTeX code, this equation is labeled ‘‘S-recursion.’’` TODO

Also, the S_i satisfy the recursion

$$S_i = 3S_{i-1} - S_{i-2}. \tag{30}$$

`In the LaTeX code, this equation is labeled ‘‘S-rec-2.’’` TODO

`In the LaTeX code, this lemma is labeled ‘‘S-formula.’’` TODO

Proof. Since $M_1 > M_0$ in any family, we must have $S_1 > S_0$, and hence the formula (29) produces only positive values of S_i. Therefore it suffices to show that (29) implies that $S_i^2 = 5M_i^2 + 4(a^2 + a - 1)$. We need to show that

$$(F_{2i}S_1 - F_{2i-2}S_0)^2 = 5(F_{2i+2}M_0 + F_{2i}c)^2 + 4(a^2 + a - 1).$$

Bringing all of the terms to the left, we need to show that the expression

$$(F_{2i}S_1 - F_{2i-2}S_0)^2 - 5(F_{2i+2}M_0 + F_{2i}c)^2 - 4(a^2 + a - 1) \tag{31}$$

`In the LaTeX code, this equation is labeled ‘‘exp.’’` TODO

is equal to zero. If we expand the squared expressions and use identities (12) and (14), along with the identities

$$\begin{aligned} S_0^2 &= 5M_0^2 + 4(a^2 + a - 1) \\ S_1^2 &= 5(3M_0 + c)^2 + 4(a^2 + a - 1), \end{aligned}$$

then (31) reduces to

$$2F_{2i}F_{2i-2}\left(15M_0^2 + 5M_0c + 6(a^2 + a - 1) - S_0S_1\right). \tag{32}$$

`In the LaTeX code, this equation is labeled ‘‘exp2.’’` TODO

In order to show that the expression (32) equals zero, it suffices to show that

$$S_0S_1 = 15M_0^2 + 5M_0c + 6(a^2 + a - 1).$$

Since both sides of this are positive, it suffices to prove that

$$S_0^2S_1^2 = (15M_0^2 + 5M_0c + 6(a^2 + a - 1))^2.$$

However, this can easily be seen by substituting the identities for S_0^2 and S_1^2, expanding both sides, and using (28). The final statement of the lemma now follows by induction, using (12). □

We find it very interesting that the M_i and S_i obey the same recurrence!

Finding the values of k Now that we have an expression for S_i, we can find all the possible values of k such that the equation $G(x) = k$ has a rational solution. Considering (24), we see that each M_i leads to two possible k-values, which we write as

$$K_i^+ = (3a - 1 + S_i)/5 \quad \text{and} \quad K_i^- = (3a - 1 - S_i)/5.$$

We need to determine when the numbers K_i^+ and K_i^- are integers. For this, we need to determine when we have

$$3a - 1 \equiv \pm S_i \pmod{5}.$$

We begin with a lemma relating the values of S_i for different i.

Lemma 10. *If $a \equiv 2 \pmod 5$, then $S_i \equiv 0 \pmod 5$ for all i. If $a \not\equiv 2 \pmod 5$, then we have $S_0 \not\equiv 0 \pmod 5$ and $S_i \equiv (-1)^i S_0 \pmod 5$ for all $i \geq 0$.*

`In the LaTeX code, this lemma is labeled ‘‘S-lem.’’` TODO

Proof. In this proof, all congruences should be interpreted modulo 5. We clearly have $S_i^2 \equiv 4(a^2 + a - 1)$ for all i. If it happens that $a \equiv 2$, then this equation becomes $S_i^2 \equiv 0$, which implies that $S_i \equiv 0$.

Suppose now that $a \not\equiv 2$. Then $S_i^2 \equiv 4(a^2 + a - 1) \not\equiv 0$, and so we see that $S_i \not\equiv 0$ for any i. If $i > 0$ then we have $S_i^2 - S_{i-1}^2 \equiv 0$, which implies that either $S_i \equiv S_{i-1}$ or $S_i \equiv -S_{i-1}$. Suppose first that $S_i \equiv S_{i-1}$. Then by (30) we have

$$S_{i+1} = 3S_i - S_{i-1} \equiv 2S_{i-1}.$$

This implies that

$$\begin{aligned} S_{i+1}^2 &\equiv 4S_{i-1}^2 \\ &\equiv 16(a^2 + a - 1) \\ &\equiv a^2 + a - 1. \end{aligned}$$

However, we know that we must have

$$S_{i+1}^2 \equiv 4(a^2 + a - 1) \not\equiv a^2 + a - 1,$$

a contradiction. Hence this case cannot occur.

Therefore we must have $S_i \equiv -S_{i-1}$ for all $i > 0$. This easily leads to having $S_i \equiv (-1)^i S_0$ for all i. This completes the proof of the lemma. □

From this lemma, we can see that if $a \equiv 2 \pmod 5$, then both K_i^+ and K_i^- are integers for all i, since we have

$$3a - 1 \pm S_i \equiv 0 \pm 0 \equiv 0 \pmod{5}.$$

For $a \not\equiv 2 \pmod 5$ and i fixed, we can see that at most one of K_i^+ and K_i^- can be an integer, since we can have either

$$3a - 1 \equiv -S_i \pmod{5} \quad \text{or} \quad 3a - 1 \equiv S_i \pmod{5},$$

but not both. Therefore, by Lemma 10, we have two possibilities in this situation. If any of K_i^+ or K_i^- are integers, then one of them is an integer exactly when i is odd and the other is an integer exactly when i is even. Otherwise, the possibility still remains that none of K_i^+ and K_i^- are integers for any values of i.

Calculating the values of x Finally, we need to find the actual values of x which guarantee that $G(x)$ is an integer when M_i is in one of the standard families. Recall that we have

$$x = \frac{-(1-a+k) \pm m}{2k},$$

where we have $m = M_i$ for some i, and k can be either K_i^+ or K_i^-. In the next two lemmas, we'll show that when K_i^+ or K_i^- is an integer, these x-values equal the ones given in the theorems.

Lemma 11. *Suppose that M_i is the family starting with $M_0 = 1$. Then K_i^+ is an integer when i is even and K_i^- is an integer when i is odd. We have*

$$\begin{aligned}
[-(1-a+K_i^+) + M_i]/2K_i^+ &= F_i/F_{i+1}, \quad i \text{ even} \\
[-(1-a+K_i^+) - M_i]/2K_i^+ &= -G_{i+1}/G_i, \quad i \text{ even} \\
[-(1-a+K_i^-) + M_i]/2K_i^- &= -G_{i+1}/G_i, \quad i \text{ odd} \\
[-(1-a+K_i^-) - M_i]/2K_i^- &= F_i/F_{i+1}, \quad i \text{ odd.}
\end{aligned}$$

If M_i is the family starting with $M_0 = a-1$, then K_i^+ is an integer when i is odd and K_i^- is an integer when i is even. We have

$$\begin{aligned}
[-(1-a+K_i^+) + M_i]/2K_i^+ &= -G_{-i}/G_{-(i+1)}, \quad i \text{ odd} \\
[-(1-a+K_i^+) - M_i]/2K_i^+ &= -F_{i+1}/F_i, \quad i \text{ odd} \\
[-(1-a+K_i^-) + M_i]/2K_i^- &= -F_{i+1}/F_i, \quad i \geq 2 \text{ even} \\
[-(1-a+K_i^-) - M_i]/2K_i^- &= -G_{-i}/G_{-(i+1)}, \quad i \geq 2 \text{ even.}
\end{aligned}$$

In the last two lines above, we cannot take $i = 0$, since $K_0^- = 0$. However, the value of x satisfying $G(x) = 0$ is $x = -a/(1-a) = -G_0/G_{-1}$.

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In the LaTeX code, this lemma is labeled ‘‘x-vals.’’
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TODO

Proof. Since the proof of each statement involves a lot of messy computations, we'll only prove the first of the 8 formulas. The proofs of the other formulas are similar. Since we are considering the family with $M_0 = 1$ and $M_1 = a + 2$, we can calculate

$$S_0 = \sqrt{5M_0^2 + 4(a^2 + a - 1)} = 2a + 1$$

and

$$S_1 = \sqrt{5M_1^2 + 4(a^2 + a - 1)} = 3a + 4.$$

We can calculate $K_0^+ = a$, and therefore we see that K_i^+ is an integer when i is even, as expected. We know then that K_i^- is an integer when i is odd.

Using the definition of K_i^+ and Lemmas 8 and 9, we can see that

$$\begin{aligned}\frac{-(1-a+K_i^+)+M_i}{2K_i^+} &= \frac{-4+2a-S_i+5M_i}{2S_i-2+6a}\\ &= \frac{[2+2F_{2i}+2F_{2i-2}]a+[-4+6F_{2i}-4F_{2i-2}]}{[6F_{2i}-4F_{2i-2}+6]a+[8F_{2i}-2F_{2i-2}-2]}.\end{aligned}$$

In order to show that this expression equals F_i/F_{i+1} when i is even, we will show that

$$\begin{aligned}&F_{i+1}\cdot([2+2F_{2i}+2F_{2i-2}]a+[-4+6F_{2i}-4F_{2i-2}])\\ &\qquad = F_i\cdot([6F_{2i}-4F_{2i-2}+6]a+[8F_{2i}-2F_{2i-2}-2])\end{aligned}$$

for even values of i. To do this, we will show that the coefficients of a on each side are equal and that the terms not multiplied by a are equal.

For the a-terms, we need to show that

$$2F_{i+1}+2F_{2i}F_{i+1}+2F_{2i-2}F_{i+1} = 6F_{2i}F_i-4F_{2i-2}F_i+6F_i.$$

That is, we need to show that the expression

$$2F_{i+1}+2F_{2i}F_{i+1}+2F_{2i-2}F_{i+1}-6F_{2i}F_i+4F_{2i-2}F_i-6F_i \tag{33}$$

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In the LaTeX code, this equation is labeled ‘‘a-exp.’’
```

TODO

is equal to zero when i is even. If we use the recurrence relation along with identities (15), (17), and (18) to write (33) in terms of only F_i and F_{i-1}, we get

$$-4F_i^3-4F_i-2F_{i-1}^3+2F_{i-1}+6F_i^2F_{i-1}+2F_iF_{i-1}^2. \tag{34}$$

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In the LaTeX code, this equation is labeled ‘‘a-exp2.’’
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TODO

However, since i is even, we have, using (10)

$$\begin{aligned}-4F_i^3-4F_i &= -4F_i(F_i^2+1)\\ &= -4F_i(F_{i+1}^2-F_{i+1}F_i)\\ &= -4F_i(F_iF_{i-1}+F_{i-1}^2).\end{aligned}$$

Similarly, using (11), we have

$$\begin{aligned}-2F_{i-1}^3+2F_{i-1} &= -2F_{i-1}(F_{i-1}^2+1)\\ &= -2F_{i-1}(F_{i-1}F_{i-2}+F_{i-2}^2)\\ &= -2F_{i-1}(F_i^2-F_iF_{i-1}).\end{aligned}$$

If we substitute these identities into the expression (34), we see that this expression really does equal zero.

To treat the terms not multiplied by a, we need to show that if i is even, then we have

$$F_{i+1}(-4+6F_{2i}-4F_{2i-2}) = F_i(8F_{2i}-2F_{2i-2}-2).$$

That is, we need to show that the expression

$$-4F_{i+1}+6F_{2i}F_{i+1}-4F_{2i-2}F_{i+1}-8F_{2i}F_i+2F_{2i-2}F_i+2F_i \tag{35}$$

is equal to zero when i is even. Again, we rewrite (35) in terms of F_i and F_{i-1}, obtaining

$$-2F_i-2F_i^3-4F_{i-1}+4F_{i-1}^3-2F_i^2F_{i-1}+6F_iF_{i-1}^2. \tag{36}$$

As with the a-terms, we can rewrite

$$-2F_i-2F_i^3 = -2F_i(F_iF_{i-1}+F_{i-1}^2)$$

and

$$-4F_{i-1}+4F_{i-1}^3 = 4F_{i-1}(F_i^2-F_iF_{i-1}).$$

Substituting these identities into (36), we see that this expression really is equal to zero for even values of i. This completes the proof of the first formula in the statement of the lemma.

As noted above, the proofs of the other formulas are all similar to this one. For the formulas involving quotients of G_i-terms, we make use of (5) and (8). When we use these formulas, the numerator and denominator of the resulting fractions each have an a^2-term in addition to the a and "constant" terms. In this case, we use the same ideas as above to prove that the coefficients of a^2 in the numerator and denominator are equal, as well as the a-coefficients and the constant terms.

To finish the proof of the lemma, we have to deal with the final case when $K_i^+ = 0$ or $K_i^- = 0$. We can see that $K_i^+ = 0$ exactly when $S_i = 1-3a$, which never occurs since S_i is defined to be positive. We have $K_i^- = 0$ exactly when $S_i = 3a-1$, which is the value of S_0 in the family starting with $M_0 = a-1$. However, when $k=0$, it is easy to see from the formula (21) for $G(x)$ that the only solution of $G(x)=0$ is $x = -a/(1-a)$, and that this is equal to $-G_0/G_{-1}$. □

To complete the proof of Theorem 1, we have by (6) that

$$-F_{i+1}/F_i = F_{-(i+1)}/F_{-i} = F_n/F_{n+1},$$

where we have set $n = -(i+1)$. Hence the four x-values involving Fibonacci numbers combine to give F_i/F_{i+1} for all $i \neq 0$. Similarly, the other four values in the lemma combine to give $-G_{i+1}/G_i$ for all $i \neq -1$. But the value with $i=-1$ is the value corresponding to $k=0$. Hence the numbers $x = -G_{i+1}/G_i$ produce integer values of $G(x)$ for all integers i. This completes the proof of Theorem 1.

To complete the proof of Theorem 2, we must deal with the remaining values of K_i^+ and K_i^-, which we have shown are integers when $a \equiv 2 \pmod 5$.

Lemma 12. *Suppose that $a \equiv 2 \pmod 5$. If M_i is the family starting with $M_0 = 1$, then we have*

$$[-(1-a+K_i^+)+M_i]/2K_i^+ = L_i/L_{i+1}, \quad i \textit{ odd}$$
$$[-(1-a+K_i^+)-M_i]/2K_i^+ = H_{-(i+1)}/H_{-i}, \quad i \textit{ odd}$$
$$[-(1-a+K_i^-)+M_i]/2K_i^- = H_{-(i+1)}/H_{-i}, \quad i \textit{ even}$$
$$[-(1-a+K_i^-)-M_i]/2K_i^- = L_i/L_{i+1}, \quad i \textit{ even,}$$

where the numbers L_i are elements of the Lucas sequence and H_i is the sequence defined in the statement of Theorem 2. If M_i is the family starting with $M_0 = a-1$, then we have

$$[-(1-a+K_i^+)+M_i]/2K_i^+ = H_i/H_{i+1}, \quad i \textit{ even}$$
$$[-(1-a+K_i^+)-M_i]/2K_i^+ = L_{-(i+1)}/L_{-i}, \quad i \textit{ even}$$
$$[-(1-a+K_i^-)+M_i]/2K_i^- = L_{-(i+1)}/L_{-i}, \quad i \textit{ odd}$$
$$[-(1-a+K_i^-)-M_i]/2K_i^- = H_i/H_{i+1}, \quad i \textit{ odd.}$$

`In the LaTeX code, this lemma is labeled ‘‘2mod5.’’` TODO

The proof of this lemma involves the same kinds of computations as in the proof of Lemma 11, and so we won't give the details here. We do note that we need not worry about having $K_0^- = 0$ since this only occurs in the family starting with $M_0 = a-1$, not the family starting with $M_0 = 1$. We can now see, using the identity (7), that the four values involving Lucas numbers combine to give L_i/L_{i+1} for all ingeters i, and the other four values combine to give H_i/H_{i+1} for all integers i. This completes the proof of Theorem 2.

Avenues for future research

`In the LaTeX code, this section is labeled ‘‘sec-future.’’` TODO

In this final section, we briefly list some possible directions for future study, which may be suitable for undergraduate research projects.

1. Can more be said about the solutions of (28)? We suspect that it should be possible to prove that there are infinitely many values of a such that there are more solutions with $1 \le m \le a-1$ than just $m = 1$ and $m = a-1$. Are there also infinitely many values of a such that $m = 1$ and $m = a-1$ are the only solutions? What if we ask the same questions, but study the cases $a \equiv 2 \pmod 5$ and $a \not\equiv 2 \pmod 5$ separately?

2. With regard to the "extra" families of solutions, is it possible to have solutions of (28) which do not lead to integer values of K_i^+ or K_i^- for any i? Or do these extra families of solutions always lead to new rational values of x which make $G(x)$ an integer? For $a \le 104$, any extra solutions do lead to new rational values of x.

3. For all of the families of solutions of $G(x) = k$ we have studied, the x-values are quotients of successive terms of a sequence satisfying the recurrence relation $A_n = A_{n-1} + A_{n-2}$. Is this still true for solutions which come from the extra families?

4. What can be said if we replace G_i^a by a different sequence?

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