Sines and Cosines of Angles in Arithmetic Progression

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In a recent math bite in this MAGAZINE [**2**], Judy Holdener gives a physical argument for the relations

$$\sum_{k=1}^{N} \cos\left(\frac{2\pi k}{N}\right) = 0 \qquad \text{and} \qquad \sum_{k=1}^{N} \sin\left(\frac{2\pi k}{N}\right) = 0,$$

and comments that "It seems that one must enter the realm of complex numbers to prove this result." In fact, these relations are consequences of more general formulas, which can be proved without using complex numbers. We state these formulas as a theorem.

THEOREM. *Suppose that $a, d \in \mathbb{R}$, $d \neq 0$, and that n is a positive integer. Then we have the formulas*

$$\sum_{k=0}^{n-1} \sin(a + kd) = \frac{\sin(nd/2)}{\sin(d/2)} \sin\left(a + \frac{(n-1)d}{2}\right)$$

and

$$\sum_{k=0}^{n-1} \cos(a + kd) = \frac{\sin(nd/2)}{\sin(d/2)} \cos\left(a + \frac{(n-1)d}{2}\right).$$

We first encountered these formulas, and also the proof given below, in the journal Arbelos, edited (and we believe almost entirely written) by Samuel Greitzer. This journal was intended to be read by talented high school students, and was published from 1982 to 1987. It appears to be somewhat difficult to obtain copies of this journal, as only a small fraction of libraries seem to hold them.

Before proving the theorem, we point out that while interesting in isolation, these formulas are more than mere curiosities. For example, the function $D_m(t) = \frac{1}{2} + \sum_{k=1}^{m} \cos(kt)$ is well-known in the study of Fourier series (see [**3**]) as the Dirichlet kernel. This function is used in the proof of Dirichlet's theorem, which implies that if a function $f(t)$ is continuous on $[-\pi, \pi]$ and has $f(-\pi) = f(\pi)$, then the Fourier series for $f(t)$ will converge to $f(t)$ at every point of $[-\pi, \pi]$. In fact, in [**3**], Pinkus and Zafrany prove that

$$D_m(t) = \frac{\sin\left(m + \frac{t}{2}\right)}{2\sin\left(\frac{t}{2}\right)}$$

using the same method as Greitzer.

Proof of the Theorem. We wish to reiterate that this proof is not original with the author. In [**1**], Greitzer gives a proof of the first formula and leaves the second as an exercise for the reader. Therefore, in this note we will prove the second formula and refer the reader to [**1**] for the proof of the first, which proceeds along exactly the same lines. First, recall the trigonometric identity

$$\sin(A + B) - \sin(A - B) = 2\cos(A)\sin(B). \tag{1}$$

Now, if we write C for the sum we wish to evaluate, then multiplying by $2\sin(d/2)$ and using (1) yields

$$\begin{aligned}
2C \cdot \sin(d/2) &= \sum_{k=0}^{n-1} 2\cos(a + kd)\sin(d/2) \\
&= \sum_{k=0}^{n-1} \sin\left(a + \frac{(2k+1)d}{2}\right) - \sin\left(a + \frac{(2k-1)d}{2}\right) \\
&= \sin\left(a + \frac{(2n-1)d}{2}\right) - \sin\left(a - \frac{d}{2}\right) \\
&= 2\cos\left(a + \frac{(n-1)d}{2}\right)\sin(nd/2),
\end{aligned}$$

where the third equality is true because the series telescopes. Solving our final equation for C completes the proof of the theorem.

References

1. S. Greitzer, Many cheerful Facts *Arbelos* **4** (1986), no. 5, 14–17.

2. J. Holdener, *Math bite: Sums of Sines and Cosines*, this MAGAZINE, **82** (2009), 126.

3. Allan Pinkus and Samy Zafrany, *Fourier Series and Integral Transforms*, Cambridge University Press, 1997.