MORE EXACT VALUES OF THE FUNCTION $\Gamma^*(k)$

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Abstract. Consider a homogeneous additive polynomial of the form $F(\mathbf{x}) = a_1x_1^k + a_2x_2^k + \cdots + a_sx_s^k$, where the coefficients are integers. The function $\Gamma^*(k)$ is defined as the smallest number s of variables such that F is guaranteed to have a nontrivial zero in every p-adic field $\mathbb{Q}_p$ regardless of the coefficients. In this article, we calculate the value of $\Gamma^*(k)$ for all $33 \leq k \leq 64$ for which this value is not already known.

1. Introduction

In this article, we study the p-adic solubility of additive forms. In particular, consider the homogeneous additive polynomial (also called a **form**)

$$F(\mathbf{x}) = a_1x_1^k + a_2x_2^k + \cdots + a_sx_s^k, \tag{1}$$

where all of the coefficients are integers. An old theorem of Brauer [3] guarantees that if the number s of variables is large enough (as a function of k), then F is guaranteed to have nontrivial zeros in every p-adic field $\mathbb{Q}_p$, regardless of the values of the coefficients. The function $\Gamma^*(k)$ gives the smallest number s that suffices to guarantee this solubility. Our goal in this article is to calculate the exact values of $\Gamma^*(k)$ for a range of values of k.

The first quantitative result on this problem was developed by Davenport & Lewis [7], who proved the upper bound $\Gamma^*(k) \leq k^2 + 1$. In the same article, they proved that this bound is sharp in the sense that $\Gamma^*(k) = k^2 + 1$ whenever $k + 1$ is prime. When $k + 1$ is composite, however, a smaller number of variables suffices. Dodson [8] has recorded the bound

$$\Gamma^*(k) \leq \frac{k^2}{2}\left(1 + \frac{2}{1+\sqrt{1+4k}}\right) + 1,$$

when $k+1$ is composite, with equality when $k = p(p-1)$ for some prime p. These, along with recent work [9] of Godinho & Knapp (which requires k to be quite large), are currently the only known formulas which give the value of $\Gamma^*(k)$ for an entire class of degrees.

Aside from these formulas, some work has been done to evaluate $\Gamma^*(k)$ for specific values of k. Lewis [15] gave the value $\Gamma^*(3) = 7$. Work of Gray [11] and Chowla [5] together shows that $\Gamma^*(5) = 16$. It appears that Bierstedt [1] was the first to prove that $\Gamma^*(7) = 22$, although this result was also independently given by Norton [16] and Dodson [8]. Norton and Dodson also independently discovered the value $\Gamma^*(9) = 37$, and Bierstedt and Norton independently gave the value $\Gamma^*(11) = 45$. Shortly afterwards, Bovey [2] showed that $\Gamma^*(8) = 39$.

After this 1974 result of Bovey, work on calculating values of $\Gamma^*(k)$ ceased for almost 40 years, possibly because the work needed to find these values requires a high amount of computation. In 2011, armed with astronomically more computing power than was available previously, Knapp computed more values of $\Gamma^*(k)$. In connection with other work, the values of $\Gamma^*(k)$ for all odd k with $13 \leq k \leq 25$ are computed in [13]. The article [12] then calculates the value of $\Gamma^*(k)$ for all $k \leq 31$ for which the value was not already known. The results of that article, when combined with the result of [14], give the value of $\Gamma^*(32)$. Finally, Jessica Kuiper (unpublished) and Daiane Veras independently discovered the value of $\Gamma^*(54)$. Veras calculated this value as part of her Ph.D. thesis, and this result has been published as one of the results of [10].

In the current article, we undertake to calculate more values of $\Gamma^*(k)$, calculating this for each $k \leq 64$ for which the value is not yet known. We thus prove the following theorem.

Theorem 1. *We have the following values of* $\Gamma^*(k)$:

$$\begin{array}{lll}
\Gamma^*(33) = 199 & \Gamma^*(44) = 177 & \Gamma^*(55) = 276 \\
\Gamma^*(34) = 205 & \Gamma^*(45) = 226 & \Gamma^*(56) = 1569 \\
\Gamma^*(35) = 211 & \Gamma^*(47) = 189 & \Gamma^*(57) = 343 \\
\Gamma^*(37) = 186 & \Gamma^*(48) = 769 & \Gamma^*(59) = 237 \\
\Gamma^*(38) = 229 & \Gamma^*(49) = 246 & \Gamma^*(61) = 306 \\
\Gamma^*(39) = 235 & \Gamma^*(50) = 351 & \Gamma^*(62) = 373 \\
\Gamma^*(41) = 247 & \Gamma^*(51) = 307 & \Gamma^*(63) = 379 \\
\Gamma^*(43) = 216 & \Gamma^*(53) = 319 & \Gamma^*(64) = 2041.
\end{array}$$

Our method to prove this theorem is largely the same as the method in [12]. Given k, we first guess the value of $\Gamma^*(k)$ by examining small primes, and then show that our guess is correct. To do this, we treat each p separately. Define $\Gamma^*(k,p)$ to be the least number of variables which guarantees that (1) has a nontrivial p-adic zero for that specific prime p, regardless of the coefficients. We then have

$$\Gamma^*(k) = \max_{p \text{ prime}} \Gamma^*(k,p),$$

and hence our goal is to show that $\Gamma^*(k,p)$ is at most the bound from the theorem for each prime p.

For each prime $p \nmid k$, we show that we are able to solve the congruence $F \equiv 0 \pmod{p}$ in such a way that there is at least one variable x_i such that $ka_i x_i^{k-1} \not\equiv 0 \pmod{p}$. Once we have this solution, we are able to use Hensel's Lemma to lift this solution of congruences to a p-adic solution of $F(\mathbf{x}) = 0$. In order to guarantee that these congruences have such "nonsingular" solutions, we use a combination of theory and computation. Appeals to a result of Dodson and to Chevalley's theorem allow us to prove the result for all but finitely many primes p. We treat almost all of these "bad" primes computationally. A result of Bovey gives us a computational method which dispenses with all but a few of these primes, and we handle the final remaining primes by a brute-force computation which essentially tests every possible congruence and verifies that there are nonsingular solutions. This leaves only the few primes for which $p|k$. For these primes, we need to study congruences modulo a power of p (which depends on k) instead of merely modulo p. We deal with these primes theoretically rather than by brute force, showing that the required p-adic zeros always exist.

Since we use the same method for each value of k, in this article we will only give a completely detailed proof for $k = 55$. We choose 55 since most of the important features of the argument appear when dealing with this value. For $k \neq 55$, we will only summarize the numerical results of our calculations when $p \nmid k$. Since we will treat the primes with $p|k$ theoretically, we will give complete proofs of our results for every k in this case. This article is structured as follows. We begin in Section 2 by showing that the values claimed in the theorem are lower bounds for the values of $\Gamma^*(k)$. In Section 3, we give the preliminary lemmas which we need to prove that the claimed values are upper bounds when $p \nmid k$. Section 4 contains the theoretical part of the proof that our theorem holds for primes with $p \nmid k$, and Section 5 contains the computational part of this proof. Finally, Section 6 completes the proof by showing that the bounds in our theorem hold when $p|k$.

As the reader has no doubt noticed, a fair portion of our work is computational. For each degree k, there are a fair number of "bad" primes that are treated computationally. To save space, we will give the numerical results for most of the relevant primes when $k = 55$, and only briefly summarize the results for other values of k. However, the authors would be happy to share both our MAPLE code and the results of our computations with any interested reader.

We finish this introduction by mentioning briefly a few ways in which our work here improves on the work in [12], where the values of $\Gamma^*(k)$ up to $k = 31$ were calculated. First, in [12], all primes with $p|k$ were treated by a brute-force method, essentially solving each possible relevant congruence. Here, the primes with $p|k$ are all treated theoretically. We feel that it is always desirable to use a theoretical rather than a computational approach when possible. Also, although it will not be apparent to the reader, we note that when we do use computational techniques, our code is better than that of [12]. While we are certain that our code can still be (perhaps greatly) improved, we have seen that our programs run faster than those in [12] and are somewhat more user-friendly. Again, the authors are happy to share our code and output with interested readers.

2. Lower Bounds

In this section, we will show that the values claimed in Theorem 1 are lower bounds for the values of $\Gamma^*(k)$. In order to do this for $k = 55$, we need to show that there exists an additive form F in 275 variables which does not have any p-adic solutions for at least one prime p. We begin the process by searching for a prime p and an additive form $F_0(\mathbf{x}) = F_0(x_1, \ldots, x_5)$ in which all of the coefficients are relatively prime to p, but for which the congruence $F_0 \equiv 0 \pmod{p}$ has no nontrivial solutions. Given such a form F_0, consider the form

$$F = F_0(\mathbf{x}_0) + pF_0(\mathbf{x}_1) + p^2F_0(\mathbf{x}_2) + \cdots + p^{54}F_0(\mathbf{x}_{54}),$$

where we write $\mathbf{x}_i = (x_{i1}, \ldots x_{i5})$, so that the variables in each occurrence of F_0 are distinct. A standard argument (see for example [16, Theorem 3.1]) shows that F has no nontrivial zeros modulo p^{55}, and hence no nontrivial p-adic zeros, Since F has 275 variables and no nontrivial p-adic zeros, this will show that $\Gamma^*(55) \geq 276$, as desired.

For most degrees k, it turns out that the most fruitful primes p to examine are those with $p \equiv 1 \pmod{k}$ and p not too large. When $k = 55$, the smallest prime with $p \equiv 1 \pmod{55}$ is $p = 331$, and a computer search shows that the congruence

$$F_0(\mathbf{x}) = x_1^{55} + 3x_2^{55} + 9x_3^{55} + 30x_4^{55} + 90x_5^{55} \equiv 0 \pmod{331}$$

has no solutions with any variable nonzero modulo 331. As discussed above, this leads to an additive form F in 275 variables with no nontrivial 331-adic zeros, as desired.

For most of the other values of k, we find lower bounds for $\Gamma^*(k)$ in the same way. We search for a prime $p \equiv 1 \pmod{k}$ and a form F_0 in t variables such that each coefficient of F_0 is relatively prime to p and such that the congruence $F_0 \equiv 0 \pmod{p}$ has only the trivial solution. As above, this immediately leads to the bound $\Gamma^*(k) \geq tk+1$. In Table 1 below, we give the prime p and the form F_0 used for each value of k. In each case, we can see that $tk + 1$ equals the value of $\Gamma^*(k)$ claimed in Theorem 1.

There are four exceptions to this method. For $k = 48$, we consider the prime $p = 17$. Since the only 48th powers modulo 17 are 0 and 1, the form $F_0 = x_1^{48} + x_2^{48} + \cdots + x_{16}^{48}$ has no nontrivial zeros modulo 17. We may then use this to create a form F in the same way as above. We treat $k = 50$ and $k = 56$ in the same way, with $p = 11$ and $p = 29$, respectively. The degree $k = 64$ is somewhat different. For this degree, we note that the form $x_1^{64} + x_2^{64} + \cdots + x_{255}^{64}$ has no nontrivial zeros modulo 2^8. From this fact, an additive form in 2040 variables with no 2-adic zeros can be constructed. We refer the reader to [14] for the details.

3. Preliminaries

As mentioned in the Introduction, our first task is to show that $\Gamma^*(k,p)$ is at most the value given in the theorem when $p \nmid k$. In this section, we give the preliminary results needed for this. Some of these results will also be useful in Section 6 when we treat the primes with $p|k$. Our first lemma combines several results of Davenport & Lewis [7].

Table 1. Forms Giving Lower Bounds For $\Gamma^*(k)$

k	p	F_0
33	67	$x_1^{33} + 2x_2^{33} + 4x_3^{33} + 8x_4^{33} + 16x_5^{33} + 32x_6^{33}$
34	103	$x_1^{34} + x_2^{34} + 25x_3^{34} + 25x_4^{34} + 48x_5^{34} + 48x_6^{34}$
35	71	$x_1^{35} + 2x_2^{35} + 4x_3^{35} + 8x_4^{35} + 16x_5^{35} + 32x_6^{35}$
37	149	$x_1^{37} + 2x_2^{37} + 4x_3^{37} + 16x_4^{37} + 63x_5^{37}$
38	191	$x_1^{38} + x_2^{38} + x_3^{38} + 143x_4^{38} + 143x_5^{38} + 143x_6^{38}$
39	79	$x_1^{39} + 2x_2^{39} + 4x_3^{39} + 9x_4^{39} + 18x_5^{39} + 36x_6^{39}$
41	83	$x_1^{41} + 2x_2^{41} + 4x_3^{41} + 8x_4^{41} + 16x_5^{41} + 32x_6^{41}$
43	173	$x_1^{43} + 2x_2^{43} + 4x_3^{43} + 32x_4^{43} + 83x_5^{43}$
44	397	$x_1^{44} + x_2^{44} + 2x_3^{44} + 8x_4^{44}$
45	181	$x_1^{45} + 2x_2^{45} + 4x_3^{45} + 8x_4^{45} + 163x_5^{45}$
47	283	$x_1^{47} + 2x_2^{47} + 4x_3^{47} + 8x_4^{47}$
48	17	$x_1^{48} + x_2^{48} + \cdots + x_{16}^{48}$
49	197	$x_1^{49} + 2x_2^{49} + 4x_3^{49} + 13x_4^{49} + 64x_5^{49}$
50	11	$x_1^{50} + x_2^{50} + \cdots + x_{10}^{50}$
51	103	$x_1^{51} + 2x_2^{51} + 4x_3^{51} + 8x_4^{51} + 16x_5^{51} + 32x_6^{51}$
53	107	$x_1^{53} + 2x_2^{53} + 4x_3^{53} + 8x_4^{53} + 16x_5^{53} + 32x_6^{53}$
55	331	$x_1^{55} + 3x_2^{55} + 9x_3^{55} + 30x_4^{55} + 90x_5^{55}$
56	29	$x_1^{56} + x_2^{56} + \cdots + x_{28}^{56}$
57	229	$x_1^{57} + 27x_2^{57} + 53x_3^{57} + 61x_4^{57} + 121x_5^{57} + 214x_6^{57}$
59	709	$x_1^{59} + 2x_2^{59} + 4x_3^{59} + 77x_4^{59}$
61	367	$x_1^{61} + 32x_2^{61} + 85x_3^{61} + 122x_4^{61} + 178x_5^{61}$
62	311	$x_1^{62} + x_2^{62} + x_3^{62} + x_4^{62} + 11x_5^{62} + 146x_6^{62}$
63	127	$x_1^{63} + 3x_2^{63} + 9x_3^{63} + 64x_4^{63} + 81x_5^{63} + 95x_6^{63}$
64	2	$x_1^{64} + x_2^{64} + \cdots + x_{255}^{64}$ (see text)

Lemma 2. *Let p be a fixed prime. In order to prove that all forms F as in* (1) *have nontrivial p-adic zeros, it suffices to consider forms with the following properties. That is, if all forms with the following properties have nontrivial p-adic zeros, then all forms as in* (1) *without these properties will also have nontrivial p-adic zeroes. We may assume*

that the form F can be rewritten as

$$F = F_0 + pF_1 + p^2F_2 + \cdots + p^{k-1}F_{k-1},$$

where the F_i are forms as in (1)*, that every variable of F appears in exactly one subform F_i, and that each coefficient of each F_i is an integer not divisible by p. Moreover, if m_i denotes the number of variables which appear in F_i with a nonzero coefficient, then we may assume that*

$$m_0 + \cdots + m_{i-1} \geq is/k \tag{2}$$

for $i = 1, 2, \ldots, k$.

If F is as in Lemma 2, a variable x which appears in the form F_i is said to be at *level i*.

Our next lemma is a standard version of Hensel's lemma, specialized for forms of the shape (1).

Lemma 3. *Let F be a form as in* (1)*. Let p be a prime, and write $k = p^\tau k_0$, where $p \nmid k_0$. Define the number γ by*

$$\gamma = \gamma(k,p) = \begin{cases} \tau + 2 & \text{if } p = 2 \text{ and } \tau \geq 1 \\ \tau + 1 & \text{otherwise.} \end{cases} \tag{3}$$

Suppose that $\mathbf{z} \in \mathbb{Z}^s$ satisfies $F(\mathbf{z}) \equiv 0 \pmod{p^\gamma}$, and that there is an integer i such that $ka_ix_i^{k-1} \not\equiv 0 \pmod{p}$. Then there is a vector $\mathbf{y} \in \mathbb{Z}_p^s$ such that $\mathbf{y} \equiv \mathbf{z} \pmod{p}$ and $F(\mathbf{y}) = 0$ in $\mathbb{Q}_p$.

Our next several lemmas give conditions under which a congruence of the form $F_0 \equiv 0 \pmod{p^t}$ is guaranteed to have solutions.

Lemma 4 (Chevalley [4])**.** *Suppose that F_0 is a form as in* (1) *in t variables, in which all coefficients are nonzero modulo p. If $t > k$, then the congruence $F_0 \equiv 0 \pmod{p}$ possesses nontrivial solutions.*

The next lemma is a trivial consequence of the proof of [8, Lemma 2.4.1].

Lemma 5 (Dodson)**.** *Suppose that F_0 is a form as in* (1) *in t variables, in which all coefficients are nonzero modulo p. Suppose that $p \nmid k$, and write $d = (k, p-1)$. If we have*

$$p > (d-1)^{(2t-2)/(t-2)}, \tag{4}$$

then the congruence $F_0 \equiv 0 \pmod{p}$ possesses nontrivial solutions.

Lemma 6 (Dodson [8])**.** *Suppose that F_0 is a form as in* (1) *in t variables, in which all coefficients are nonzero modulo p, and suppose that -1 is a k-th power modulo p^γ. If we have $2^t > p^\gamma$, then the congruence $F_0 \equiv 0 \pmod{p^\gamma}$ possesses a nontrivial solution, i.e., a solution with at least one variable not divisible by p.*

Lemma 7 (Chowla, Mann, Straus [6])**.** *Suppose that $(k, p-1) < (p-1)/2$ and that $n \geq \frac{(k,p-1)+1}{2}$. Then the form $a_1x_1^k + \cdots + a_nx_n^k$, with all coefficients nonzero modulo p, represents every residue modulo p.*

Note that in this lemma, it is possible that the only representation of the zero residue is trivial. It is easy to see that if we have a single additional variable, then the zero residue has a nontrivial representation, obtained by setting $x_1 = 1$ and using the other variables to represent $-a_1 \pmod{p}$. Hence we have the following trivial corollary.

Corollary 8. *Suppose that $(k, p-1) < (p-1)/2$ and that $n \geq \frac{(k,p-1)+3}{2}$. Then the form $a_1x_1^k + \cdots + a_nx_n^k$, with all coefficients nonzero modulo p, represents every residue modulo p nontrivially. In particular, the congruence $a_1x_1^k + \cdots + a_nx_n^k \equiv 0 \pmod{p}$ has a nontrivial solution.*

Lemma 9 (Bovey [2])**.** *Suppose that k, p, t are positive integers with p prime and $p \nmid k$. Define the function $Q(k, p, t)$ by*

$$Q(k,p,t) = \frac{1}{p^t - p} \sum_{b=1}^{p-1} |S(b)|^t,$$

where we have

$$S(b) = \sum_{x=0}^{p-1} e_p(bx^k), \qquad \textit{and} \qquad e_p(x) = e^{2\pi i x/p}.$$

If $Q(k,p,t) < 1$, then every congruence

$$F_0 = a_1x_1^k + a_2x_2^k + \cdots + a_tx_t^k \equiv 0 \pmod{p},$$

with all coefficients nonzero modulo p, has a nontrivial solution.

As Bovey points out, according to Dodson [8, Lemma 4.2.1], this result implies that if $Q(k,p,t) < 1$, then $\Gamma^*(k,p) \leq k(t-1)+1$.

4. Large Primes

In this section, we prove that the value of $\Gamma^*(k,p)$ is at most the value given in the theorem whenever p is sufficiently large. For each degree k, this will leave us with only a finite number of primes which require further analysis. Our main tool in this section is Lemma 5. As in Section 2, we give the full details only for $k = 55$. For this degree, we need to show that any additive form in 276 variables must have a nontrivial p-adic zero for any sufficiently large prime p. Suppose that p is a prime such that $p \nmid 55$ and let F_0 be defined as in Lemma 2. Then by Lemma 2 we may write (renaming some of the variables if necessary)

$$F_0 = a_1x_1^{55} + \cdots + a_tx_t^{55}, \tag{5}$$

where $t \geq 6$ and each coefficient of F_0 is relatively prime to p. If we can show that the congruence $F_0 \equiv 0 \pmod{p}$ has a solution in which at least one variable is nonzero modulo p, then Hensel's lemma (Lemma 3) allows us to lift this solution to a solution of $F = 0$ in $\mathbb{Q}_p$.

First, suppose that p is a prime with $p \nmid 55$ and write $d = (55, p-1)$. Then the set of 55-th powers modulo p is the same as the set of d-th powers modulo p. Hence the congruence $F_0 \equiv 0 \pmod{p}$ has a nontrivial solution if and only if the congruence

$$a_1x_1^d + \cdots + a_tx_t^d \equiv 0 \pmod{p}$$

does. Suppose first that $d \in \{1, 5\}$. Since $t > 5$, Chevalley's theorem (Lemma 4) says that this latter congruence does have nontrivial solutions. This finishes the proof for these primes.

Therefore we need only concern ourselves with primes p such that $d = 11$ or $d = 55$. If $d = 11$, then Lemma 5, with $t = 6$, shows that we can solve the congruence $F_0 \equiv 0 \pmod{p}$ nontrivially whenever $p > 10^{(10/4)} \approx 316.2$. If $d = 55$, then Lemma 5 tells us that we are done whenever $p > 54^{10/4} \approx 21428.1$. Therefore, when $k = 55$ we only need to treat primes such that one of the following conditions holds:

- $p|55$
- $(55, p-1) = 11$ and $p \leq 316$
- $(55, p-1) = 55$ and $p \leq 21428$.

In Table 2 below, we summarize the results of applying this analysis to each of our degrees. To read the tables, k is the degree under consideration, as usual, and t represents the number of variables which we can assume to exist in F_0. In the third column, d represents the possible values of $d = (k, p-1)$. We ignore the possibility that $d < t$, as in these cases we may simply use Chevalley's theorem as above. For each value of d, the last column gives the result of applying Lemma 5. Thus, for each value of k, the only primes $p \nmid k$ we need to further consider are the ones for which $(k, p-1)$ is one of the values in the table and p is at most the number in the last column.

5. The Remaining Primes with $p \nmid k$

In this section, we treat the remaining primes with $p \nmid k$. When $k = 55$, we need to treat the primes with $(55, p-1) = 55$ and $p \leq 21428$, and the primes with $(55, p-1) = 11$ and $p \leq 316$. We begin the study of these primes by using MAPLE to calculate the function $Q(55, p, 6)$ defined in Lemma 9. For each prime for which $Q(55, p, 6) < 1$, we know that the congruence $F_0 \equiv 0 \pmod{p}$ has nontrivial solutions. As above, Hensel's lemma then shows that the equation $F = 0$ has a nontrivial p-adic solution. In the tables below, we give the values of this function (rounded to three significant decimal places) for all relevant primes p with $(55, p-1) = 11$ and all the primes up to 4951 with $(55, p-1) = 55$. We have omitted the values of $Q(55, p, 6)$ for the

TABLE 2. Bounds From Lemma 5

k	t	d	Bound
33	7	11	251.2
		33	4096
34	7	34	4409.9
35	7	7	73.7
		35	4737.5
37	6	37	7776
38	7	38	5803.4
39	7	13	389.1
		39	6187.008
41	7	41	6997.5
43	6	43	11432.0
44	23	44	2645.5
45	6	9	181.02
		15	733.4
		45	12842.0
47	5	47	27165.7
48	17	24	803.6
		48	3691.0

k	t	d	Bound
49	6	7	88.2
		49	15962.6
50	11	50	5701.5
51	7	17	776.0
		51	11954.4
53	7	53	13134.3
55	6	11	316
		55	21428
56	29	56	4070.4
57	7	19	1029.6
		57	15691.0
59	5	59	50405.1
61	6	61	27885.5
62	7	62	19266.0
63	7	7	73.7
		9	147.0
		21	1325.8
		63	20032.7
64	32	64	5231.6

primes with $4952 \leq p \leq 21428$ and $(55, p-1) = 55$ due to a lack of space, but assure the reader that these values are all less than 1.

TABLE 3. Values of $Q(55, p, 6)$ when $(55, p-1) = 11$

p	$Q(55, p, 6)$
23	4.710
67	1.949
89	0.811
199	0.174

TABLE 4. Values of $Q(55, p, 6)$ when $(55, p-1) = 55$

p	$Q(55,p,6)$	p	$Q(55,p,6)$
331	13.489	2861	0.169
661	4.133	2971	0.0894
881	2.088	3191	0.258
991	1.760	3301	0.201
1321	0.884	3631	0.416
1871	0.680	3851	0.118
2311	0.209	4621	0.0872
2531	0.209	4951	0.0484

At this point, we have dealt with all primes $p \nmid 55$ except for the six for which $Q(55, p, 6) > 1$. For the prime $p = 23$, Lemma 6 shows that having 6 variables at level 0 suffices to guarantee solubility. For the rest of the primes, we use MAPLE to make a brute-force computation, showing that all possible congruences $F_0 \equiv 0 \pmod{p}$ have solutions. However, we do not actually need to check all $(p-1)^6$ possible forms F_0. It suffices to check a much smaller subset, as we now show. For simplicity, we will work with $p = 331$, but it is easy to see that these ideas always suffice. First, by multiplying F_0 by $a_1^{-1} \pmod{331}$, we may assume that $a_1 = 1$. Now let g be a primitive root modulo 331. If a is one of the coefficients in F_0, we can write $a = g^b$ with $0 \le b \le 330$. We can further write $b = 55q + r$, with $0 \le q \le 6$ and $0 \le r \le 54$. Then we can rewrite any term ax^{55} in F_0 as

$$ax^{55} = g^{55q+r}x^{55} = g^r(g^q x)^{55}.$$

If we then make the change of variables $y = g^q x$, we see that we may assume that every coefficient of F_0 has the form g^r for some $r = 0, 1, \ldots, 54$. Finally, after making these changes, we may relabel the variables so that the exponents are a (weakly) increasing sequence. We then use MAPLE to verify that all of the forms F_0 in 5 variables meeting these requirements have nontrivial zeros modulo 331. To get an idea of how much we're economizing here, we note that for $p = 331$ we need to find nontrivial zeros of 5006386 congruences instead of $330^6 \approx 1.29 \times 10^{15}$ congruences. MAPLE's computations show that every possible congruence does in fact have a nontrivial solution, and so we are finished with these primes.

For the other degrees, we follow the same procedure. In Table 5 below, we give the primes for each k which require further testing since $Q(k,p,t) \geq 1$. For the primes p with an asterisk as a superscript, Lemma 6 shows that all additive congruences of degree k in t variables have nontrivial solutions modulo p. For the one prime with a double asterisk, Corollary 8 shows that the congruences all have solutions. The other primes were all tested using the MAPLE routine. In each case, MAPLE verified that all the relevant congruences do in fact have nontrivial solutions modulo p. This finishes the testing for these primes.

6. Primes Dividing the Degree

At this point, we have dealt with all of the primes for which $p \nmid k$, and have shown that for these primes, the number of variables given in the theorem suffices to guarantee p-adic solubility. We now finish our proof by dealing with the primes for which $p|k$. We could do this through more brute force calculations, but since it is possible to treat these primes theoretically, we prefer to do so.

In this section, it makes less sense than in the remainder of the article to treat the degree $k = 55$ completely before treating the other degrees. Although the proofs of the required lemmas are similar, we need to use different lemmas for different degrees, and the work for other degrees does not merely repeat our work for $k = 55$. Therefore, we merely state for now that Corollary 14 below shows that $\Gamma^*(55,5) \leq 83$ and Lemma 19 shows that $\Gamma^*(55,11) \leq 276$. Combined with our previous work, this shows that $\Gamma^*(55,p) \leq 276$ for every prime p, completing the proof that $\Gamma^*(55) = 276$.

Before giving the lemmas we need to complete the proof, we briefly summarize the results of our work. In Table 6, we list each pair k,p that we need to treat in this section and state both our upper bound

TABLE 5. Primes with $Q(k, p, t) \geq 1$

k	t	d	Primes
33	7	11	23^*
		33	67, 199, 331 397
34	7	34	103, 137, 239 409
35	7	7	29^*
		35	71^*, 211, 281 421
37	6	37	149, 223, 593
38	7	38	191, 229, 457
39	7	13	53^*
		39	79^*, 157, 313 547
41	7	41	83^*, 739
43	6	43	173, 431
44	23	44	89
45	6	9	19^*, 37^*
		15	31^*, 61^*
		45	181, 271, 541, 631, 811, 991, 1171
47	5	47	283, 659, 941 1129, 1223, 1693 1787
48	17	24	73^{**}
		48	97^*, 193

k	t	d	Primes
49	6	7	29
		49	197, 491, 883
50	11	50	101, 151, 251 401
51	7	17	137
		51	103^*, 307, 409 613
53	7	53	107^*, 743, 1061
55	6	11	23^*, 67, 89, 199
		55	331, 661, 881 991
56	29	56	113^*, 337
57	7	19	none
		57	229, 457, 571
59	5	59	709, 827, 1063 1181, 1889, 2243
61	6	61	367, 733, 977 1709
62	7	62	311, 373, 683 1117
63	7	7	29
		9	19^*, 37
		21	43^*
		63	127^*, 379, 631 757, 883
64	32	64	193, 257

on $\Gamma^*(k, p)$ and which lemma below gives this bound. We note that these are upper bounds only, and do not claim that these are the exact values of $\Gamma^*(k, p)$. However, these bounds are all at most the values of $\Gamma^*(k)$ given in the theorem. Combined with our previous work, this completes the proof of the theorem.

TABLE 6. Bounds on $\Gamma^*(k,p)$ when $p|k$

k	p	$\Gamma^*(k,p) \leq$	Source
33	3	100	Lem. 11
	11	199	Lem. 11
34	2	79	Lem. 12
	17	137	Cor. 16
35	5	141	Lem. 11
	7	176	Lem. 11
37	37	56	Cor. 14
38	2	88	Lem. 12
	19	153	Cor. 16
39	3	118	Lem. 11
	13	196	Cor. 16
41	41	62	Cor. 14
43	43	65	Cor. 14
44	2	166	Lem. 12
	11	177	Cor. 16
45	3	136	Cor. 18
	5	68	Cor. 14
47	47	71	Cor. 14
48	2	505	Lem. 12
	3	193	Lem. 12

k	p	$\Gamma^*(k,p) \leq$	Source
49	7	148	Cor. 18
50	2	116	Lem. 12
	5	301	Lem. 11
51	3	203	Lem. 12
	17	77	Cor. 14
53	53	80	Cor. 14
55	5	83	Cor. 14
	11	276	Lem. 19
56	2	343	Lem. 12
	7	1345	Lem. 12
57	3	227	Lem. 12
	19	286	Cor. 16
59	59	89	Cor. 14
61	61	92	Cor. 14
62	2	144	Lem. 12
	31	249	Cor. 16
63	3	190	Cor. 18
	7	316	Cor. 16
64	2	2041	Lem. 12

We begin with a few lemmas about solutions of congruences modulo a prime which supplement those given in Section 3.

Lemma 10 (Davenport & Lewis [7]). *Suppose that F_0 is a form as in* (1) *in t variables, in which all coefficients are nonzero modulo p. If $t > (k, p-1)$, then the congruence $F_0 \equiv 0 \pmod{p}$ possesses nontrivial solutions. Moreover, if we specify any variable in advance, we can find* *a solution of the congruence in which the specified variable is nonzero modulo p.*

As a trivial consequence of Lemma 6, we can prove a bound on $\Gamma^*(k,p)$ which is frequently useful.

Lemma 11. *If k and p are as in Lemma 6, then we have $\Gamma^*(k,p) \leq k \cdot \left[\frac{\gamma \log p}{\log 2}\right] + 1$, where $[\cdot]$ is the greatest integer function.*

Proof. Suppose that F has s variables, where $s \geq k \cdot \left[\frac{\gamma \log p}{\log 2}\right] + 1$. Then by Lemma 2, the form F_0 contains t variables, where

$$t \geq \frac{s}{k} \geq \left[\frac{\gamma \log p}{\log 2}\right] + \frac{1}{k} > \left[\frac{\gamma \log p}{\log 2}\right].$$

Since t is an integer, this guarantees that $t > \gamma \log p / \log 2$. Hence $2^t > p^\gamma$, and so the congruence $F_0 \equiv 0 \pmod{p^\gamma}$ possesses a nontrivial solution by Lemma 6. This lifts to a p-adic solution of $F = 0$ by Lemma 3. □

Lemma 12 (Godinho, et. al. [10])**.** *Suppose that $k \in \mathbb{Z}^+$ and that p is a prime. Write $k = p^\tau k_0$ with $p \nmid k_0$ and define the number γ as in (3). Finally, write $k = \gamma q + r$ with $q, r \in \mathbb{Z}$, $0 \leq r < \gamma$. Then we have*

$$\Gamma^*(k,p) \leq (p^\gamma - 1)q + p^r,$$

and equality holds whenever $p - 1$ divides k.

We now begin to prove some lemmas that give the values of $\Gamma^*(k,p)$ for various choices of k and p. We prove these lemmas through the well-known method of contractions, which we now briefly describe. Recall that in the notation of Lemma 2, a variable of F that appears in the subform F_i is said to be at level i. Suppose that a portion of F at level i looks like $p^i(a_1 x_1^k + \cdots + a_n x_n^k)$ and that we can find numbers $b_1, \ldots, b_n$ not all zero modulo p, such that $a_1 b_1^k + \cdots + a_n b_n^k \equiv 0 \pmod{p}$. If we then set $x_i = b_i T$ for $1 \leq i \leq n$, where T is a new variable, then T is at a level strictly higher than i. We refer to this as making a contraction of variables from level i to a higher level.

Now, define the number γ as in Lemma 3, and suppose that we can use a variable at level 0 in a contraction, or a series of contractions, which results in a variable T at level at least γ. If we set $T = 1$ and all other variables equal to 0, then we have a solution of $F \equiv 0 \pmod{p^\gamma}$. Moreover, if we trace T back to the original variables, then this solution nontrivially involves a variable at level 0. Thus this solution is nonsingular and lifts to a p-adic solution by Lemma 3. In the proofs below, we refer to a variable at level 0 as *primary*, and also apply this term to any variable which has been created by contractions involving a primary variable. All other variables will be called *secondary*. This discussion shows that if we can use contractions to create a primary variable at level γ, then we are able to find a p-adic zero of F.

We may now begin proving some upper bounds on values of $\Gamma^*(k, p)$, where p divides k. To make these lemmas a bit easier to state and prove, we define the number $\gamma^*(k, p)$ to be the smallest number t of variables which guarantees that the congruence

$$a_1x_1^k + \cdots + a_tx_t^k \equiv 0 \pmod{p}$$

always has nontrivial solutions regardless of the coefficients. For example, in this language, Lemma 10 states that $\gamma^*(k, p) \le (k, p-1) + 1$.

Lemma 13. *Suppose that p is an odd prime, that $k = pk_0$ with $p \nmid k_0$, and that $\gamma^*(k, p) = 2$. Then we have $\Gamma^*(k, p) \le \frac{3}{2}k + 1$.*

Proof. We need to show that we can construct a primary variable at level 2. By Lemma 2, we may assume that $m_0 \ge 2$ and that $m_0 + m_1 \ge 4$. Suppose first that $m_0 \ge 4$. Since $\gamma^*(k, p) = 2$, we may use contractions to construct two primary variables at level 1 or higher. If we do not already have a primary variable at level 2, then we have two primary variables at level 1, which can be contracted to a primary variable at level 2, and we are done. Otherwise, we have $m_0 \ge 2$ and $m_1 \ge 1$. We can use two variables at level 0 to construct a primary variable at level 1 or higher. If it is at level exactly 1, then we can contract it with a secondary variable at this level to make a primary variable at level at least 2. This completes the proof of the lemma. □

Corollary 14. *Suppose that p is an odd prime and that $k = pk_0$, where $p \nmid k_0$. If $(k, p-1) = 1$, then we have $\Gamma^*(k,p) \leq \frac{3}{2}k + 1$.*

Proof. The assumption $(k, p-1) = 1$ implies that $\gamma^*(k,p) = 2$ by Lemma 10. Then Lemma 13 completes the proof. □

Lemma 15. *Suppose that p is an odd prime and that $k = pk_0$ with $p \nmid k_0$.*
a) If $\gamma^*(k,p) = 3$, then $\Gamma^*(k,p) \leq 5k + 1$.
b) If $(k, p-1) = 2$, then $\Gamma^*(k,p) \leq 4k + 1$.

Proof. We prove part (a) first. Suppose that we have $s = 5k + 1$ variables. By Lemma 2, we have $m_0 \geq 6$ and $m_0 + m_1 \geq 11$. We need to show that we can create a primary variable at level 2. If it happens that $m_0 \geq 9$, then we can contract the variables at level 0 to three primary variables at level at least 1. If they are all at level 1, then we may contract them to form a primary variable at level at least 2, and we are done. If $6 \leq m_0 \leq 8$, then we can create two primary variables at level at least 1. If they are both at level exactly 1, then we can use them, along with a secondary variable at level 1, to make a primary variable at level at least 2, and we are done. This completes the proof of this case.

Now we prove part (b). Note that the condition $(k, p-1) = 2$ implies that $\gamma^*(k,p) \leq 3$ by Lemma 10. Lemma 2 gives us $m_0 \geq 5$ and $m_0 + m_1 \geq 9$. If $m_0 \geq 6$, then we may proceed exactly as in part (a). (Note that in the proof of part (a), we only needed to know that $m_0 + m_1 \geq 9$.) If $m_0 = 5$ then we can create a primary variable at level at least 1. If this variable is at level exactly 1, then Lemma 10 guarantees that we can use this primary variable in a contraction with two secondary variables at level 1 to produce a primary variable at level at least 2. (Note that in the proof of part (a), we were not able to guarantee that the primary variable could be used in this contraction.) This completes the proof of the lemma. □

Corollary 16. *We have the following upper bounds:*

$$\begin{array}{lll} \Gamma^*(39,13) \leq 196 & \Gamma^*(57,19) \leq 286 & \Gamma^*(63,7) \leq 316 \\ \Gamma^*(34,17) \leq 137 & \Gamma^*(38,19) \leq 153 & \\ \Gamma^*(44,11) \leq 177 & \Gamma^*(62,31) \leq 249. & \end{array}$$

Proof. First, note that we have $\gamma^*(39,13) = \gamma^*(57,19) = 3$ by Corollary 8. Moreover, since $x^{63} \equiv x^3 \pmod 7$, we have $\gamma^*(63,7) = \gamma^*(3,7) \leq 3$ by Lemma 6. The values in the first row now follow from part (a) of Lemma 15. The rest of the values follow immediately from part (b) of Lemma 15. ☐

Lemma 17. *Suppose that p is an odd prime and that $k = p^2 k_0$ with $p \nmid k_0$. Suppose further that $\gamma^*(p,k) = 2$. Then we have $\Gamma^*(k,p) \leq 3k+1$.*

Proof. We need to show that it is possible to construct a primary variable at level at least 3. By Lemma 2, we have $m_0 \geq 4$, $m_0 + m_1 \geq 7$, and $m_0 + m_1 + m_2 \geq 10$. Suppose first that $m_0 \geq 8$. Then we can contract the variables at level 0 to form 4 primary variables at level at least 1. From these, we can make 2 primary variables at level at least 2, and hence at least one primary variable at level at least 3.

If $6 \leq m_0 \leq 7$, then we can contract the variables at level 0 to form 3 primary variables at higher levels, and we can assume that they are at level at most 2. If at least two of them are at level 2, then we can contract them to a primary variable at level 3, and we are done. If exactly one of them is at level 2, then we contract the two primary variables at level 1 to a primary variable at level at least 2. Then if necessary, we contract two primary variables at level 2, and we are done. Finally, if we have three primary variables at level 1, then we contract two of them to level (at least) 2. We have one secondary variable at level either 1 or 2. If it is at level 1, then we may contract it with the remaining primary variable there to produce a primary variable at level at least 2, and then if necessary contract the two primary variables at level 2. Otherwise, the secondary variable is at level 2, and we contract it with the primary variable there to produce a primary variable at level at least 3.

If $4 \leq m_0 \leq 5$, then we can create two primary variables at levels at least 1, and we can assume that they are at level at most 2. If both are at level 2, then we contract them and are done. Suppose that there is one primary variable at each level. We contract the primary variable at level 1 with a secondary variable there to create a primary variable at level at least 2. If necessary, we then contract that with the other primary variable at level 2, and we are done. Finally, if both primary variables are at level 1, then we contract each with a secondary variable there to produce two primary variables at higher levels. If they are both at level 2, then we contract them, completing the proof. □

Corollary 18. *We have the following values:*

$$\Gamma^*(45,3) \leq 136 \qquad \Gamma^*(49,7) \leq 148 \qquad \Gamma^*(63,3) \leq 190$$

Proof. In each of these cases, we have $(k, p-1) = 1$, so that $\gamma^*(k,p) = 2$ by Lemma 10. The bounds then follow from Lemma 17. □

With the above lemmas, we can find upper bounds for $\Gamma^*(k,p)$ in all the cases we need except for $\Gamma^*(55,11)$, which we treat now. With this final lemma, the proof of the theorem is complete.

Lemma 19. *We have* $\Gamma^*(55,11) \leq 276$.

Proof. Suppose that we have $s = 276$ variables. We need to show that we can construct a primary variable at level 2. By Lemma 2, we have $m_0 \geq 6$ and $m_0 + m_1 \geq 11$. Since $x^{55} \equiv x^5 \pmod{11}$ for all x, we have $\gamma^*(55,11) = \gamma^*(5,11)$. Lemma 6 with $\gamma = 1$ then gives us[1] $\gamma^*(55,11) \leq 4$. However, if there are at least 6 variables at level 0, then there must be two whose coefficients are either equal or negatives modulo 11, and these two variables can be contracted to a variable at a higher level. We now proceed as follows. If $m_0 \geq 10$, then we can contract the variables at level 0 to form at least 4 variables at higher levels. If any of these are at level at least 2, then we are done. Otherwise, we have four primary variables at level 1, and these can be

[1]This special case of Lemma 6 is also part of [8, Lemma 2.2.1]. That lemma in fact shows that $\gamma^*(5,11) = 4$.

contracted to a primary variable at a higher level, and we are done. Next, if $8 \leq m_0 \leq 9$, then we can use contractions to make at least 3 primary variables at level 1, and we know that there are at least 2 other variables already there. Using the three primary variables and one of the secondary variables, we can make a contraction to a higher level, and the resulting new variable must be primary. Finally, if $6 \leq m_0 \leq 7$, then we can create at least 2 primary variables at level 1, and since $m_1 \geq 4$, there are at least four secondary variables already there. Since we have $6 > 5 = (55, 10)$ variables at level 1, Lemma 10 guarantees that we can use one of the primary variables in a contraction to a higher level. Thus, regardless of the value of m_0, we can create a primary variable at level at least 2, and therefore can find an 11-adic zero of F. This completes the proof of the lemma.

□

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