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Diagonal forms over quadratic extensions of $\mathbb{Q}_2$

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Abstract. In 1920, Emil Artin conjectured: Let K be a field complete with respect to a discrete absolute value, with finite residue field. Then every homogeneous form with coefficients in K and degree d with at least $d^2 + 1$ variables admits a non-trivial zero. In this article we prove the conjecture for diagonal forms of degree d not power of 2 over any quadratic extension of $\mathbb{Q}_2$.

1. Introduction

Let K be a field complete with respect to a discrete absolute value and with finite residue field. We denote by O_K the set of the integers of K. Consider a diagonal form

$$\mathscr{F} \;=\; a_1x_1^d + \cdots + a_Nx_N^d \tag{1}$$

of degree d in N variables and with coefficients in K. We define $\Gamma^*(d, K)$ as the least positive integer such that, whenever $N \geq \Gamma^*(d, K)$, then there exists a non-trivial zero for $\mathscr{F}$. In 1920, the Austrian mathematician Emil Artin conjectured:

Conjecture. *Let K be a field complete with respect to a discrete absolute value and with finite residue field. If $\mathscr{F}$ is a homogeneous form of degree d in N variables and coefficients in K, then $N \geq d^2 + 1$ guarantees the existence of a non-trivial zero for $\mathscr{F}$.*

Lang [5] proved that the conjecture is true when K is a field of power series over a finite field. However, in 1966, Terjanian [9] found a homogeneous form of degree 4 in 18 variables and coefficients in $\mathbb{Q}_2$ with no non-trivial zeros.

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However, no counterexample has yet been found if we look only for diagonal forms. Furthermore, in 1963, Davenport and Lewis [3] proved the following version of the conjecture. For every $d \in \mathbb{N}$ we have $\Gamma^*(d, \mathbb{Q}_p) \leq d^2 + 1$. Moreover, if $d + 1 = p$, then $\Gamma^*(d, \mathbb{Q}_p) = d^2 + 1$. Since then, a lot of research has been done in order to generalize this result to finite extensions of $\mathbb{Q}_p$. In 2008, Brink, Godinho and Rodrigues [2] proved that if K is a finite extension of $\mathbb{Q}_p$ of degree n and $d = p^l \cdot m$ with $(m, p) = 1$, then

$$\Gamma^*(d, K) \leq d^{2l+5} + 1$$

and

$$\Gamma^*(d, K) \leq 4nd^2 + 1.$$

The second estimate above gives one hope to prove Artin's conjecture for diagonal forms over finite extensions of $\mathbb{Q}_p$. In recent years, there have been several significant results on this problem. In 2017, Moore [6] showed that

$$\Gamma^*(d, K) \leq \begin{cases} d(md+1)^{l+1} & p > 2 \\ d(md+1)^{l+2} & p = 2, \end{cases}$$

and recently Skinner [7] has improved this to

$$\Gamma^*(d, K) \leq \begin{cases} md\left(\frac{p^{l+1}-1}{p-2}\right) & p > 2 \\ 2md(p^{l+2} - 1) & p = 2. \end{cases}$$

We note that the results of Moore and Skinner just quoted are both specializations of results which apply to systems of diagonal forms. Additionally, Vieira proved in his Ph.D. thesis [8] that if $p > 2$ and the extension $K/\mathbb{Q}_p$ is unramified, then

$$\Gamma^*(d, K) \leq d^2 + 1,$$

and Knapp [4] has shown that if K is any ramified quadratic extension of $\mathbb{Q}_2$, then $\Gamma^*(6, K) \leq 9$. In this article we prove the following result:

Theorem 1. *Let K be any quadratic extension of $\mathbb{Q}_2$. For d not a power of 2 we have $\Gamma^*(d, K) \leq d^2 + 1$.*

In the proof of Theorem 1, we will take a diagonal form $\mathscr{F}$ of degree d in $N = d^2 + 1$ variables, with coefficients in O_K (this can be assumed since K is the field of fractions of O_K), and find non-singular solutions for $\mathscr{F}$ modulo a specific power of π (the uniformizer of K). Then we can, by means of a version of Hensel's Lemma, lift these solutions to non-trivial solutions of $\mathscr{F} = 0$. To find these non-singular solutions, we will use the method of contractions, which, roughly speaking, involves solving congruences one power of π at a time until we can achieve a solution modulo a specific power of π. To better work with the aforementioned contractions we will make use of the fact that every element of O_K can be written as a power series in π and then obtain some combinatorial results.

Once we have developed these preliminary results, there are unfortunately numerous cases that must be considered. However, in all of these cases, we use the same ideas in the proof. Our goal in each case is to use contractions to produce a "primary" variable at a sufficiently high "level" of $\mathscr{F}$ (these terms are defined in Section 2). We begin by creating as many primary variables as possible by making contractions of variables at "level 0." If K is the unramified extension of $\mathbb{Q}_2$, then we find ourselves in one of two situations. If sufficiently many of these primary variables are at a high enough level, then our goal is to use these primaries in our final contractions. In order to do this, we need some additional variables at these high levels, and we show that that we are able to make contractions of the "secondary" (i.e., non-primary) variables to obtain these. On the other hand, if our initial contractions do not yield enough primary variables at high levels, then we are able to make a change of variables such that the form $\mathscr{F}'$ produced by this change has more variables at low levels than $\mathscr{F}$ does. We then begin the problem anew, using $\mathscr{F}'$ instead of $\mathscr{F}$. After at most two iterations of this process, we are able to produce the primary variables at high levels that we need to finish the proof.

If the extension K is ramified, it turns out that using the initial form $\mathscr{F}$, we are always able to get primary variables at high enough levels, so that we do not need to make changes of variables as above. However, there is a different issue which arises for these extensions. In the step where secondary variables are contracted, the contractions lift variables by 2 levels at a time, and therefore we may not be able to create secondary variables at exactly the same levels as the primary variables. If this is the case, then instead of using the variables at level 0 to create the primaries, we will use the variables at level 1. We can then show that with this new type of primary variable, we are able to create primaries at the same high level as the secondaries, allowing us to complete the proof.

To finish this introduction, it is appropriate to comment briefly on the quality of these results. As mentioned above, the bound $\Gamma^*(d, K) \leq d^2 + 1$ is sharp when $K = \mathbb{Q}_p$. Philosophically, the problem should be at least as difficult for extension fields as it is for $\mathbb{Q}_p$, since one needs to find nonsingular solutions modulo higher powers of π in order to use Hensel's Lemma. Therefore one suspects that d^2+1 is the best upper bound that can be hoped for in these cases. However, we do not know of any specific example of an extension field K and a form of degree d in d^2 variables that has no nontrivial zeros over K. We also note that our theorem does not include the case where d is a power of 2, although we conjecture that the d^2+1 bound holds for these degrees as well. For these degrees, we find that the number of variables at level 0 is insufficient for our contractions to produce primary variables at high enough levels to use Hensel's Lemma. It is likely that a significantly more delicate analysis would be required to prove Artin's conjecture for these degrees.

The remainder of this article is divided into 4 sections. In the next section we present some basic concepts regarding finite extensions of $\mathbb{Q}_p$ and some techniques that we will use in the proof of Theorem 1. In sections 3 and 4 we prove a series of combinatorial lemmas for elements of O_K, where K is a quadratic extension (unramified - section 3, ramified - section 4) of $\mathbb{Q}_2$. In section 5 we present the proof of Theorem 1.

2. Preliminary Concepts

In this section we introduce notation and ideas that will be used throughout the proof of Theorem 1. We set K as a quadratic extension of $\mathbb{Q}_2$ and take π a uniformizer of K. Then every integer $a \in O_K$ can be written as

$$a = \sum_{i \geq 0} a_i \cdot \pi^i \tag{2}$$

where the a_i belong to a set $\mathscr{R}$ of representatives of the residue field $k_K = O_K/(\pi)$. For the six ramified extensions

$$\mathbb{Q}(\sqrt{\pm 2}),\ \mathbb{Q}(\sqrt{\pm 10}),\ \mathbb{Q}(\sqrt{-1}),\ \text{and}\ \mathbb{Q}(\sqrt{-5})$$

we have $2 \equiv \pi^2 (\text{mod} \pi^3)$ and $k_K = \mathbb{F}_2$ so we can take $\mathscr{R} = \{0, 1\}$. For the unramified extension $\mathbb{Q}(\sqrt{5})$ we have $2 = \pi$ and $k_K = \mathbb{F}_4$ so we take $\mathscr{R} = \{0, 1, \alpha, \alpha + 1\}$.

If $b \in O_K$ is such that

$$b = \pi^k (b_0 + b_1 \pi + b_2 \pi^2 + \cdots) \tag{3}$$

where $b_i \in \mathscr{R}$ for all $i \geq 0$ and $b_0 \neq 0$, we say that b is an element at **level** k and that b_0 is the **zeroterm** of b. Extending this notion, we say that a variable x_j of the form $\mathscr{F}$ (see (1)) is at **level** k and has zeroterm b_0 if its coefficient a_j does.

Davenport and Lewis [3] defined an equivalence relation on the set of additive forms that allows us to work with a special type of form. While they worked with diagonal forms over $\mathbb{Q}_p$, their proof can be easily adapted to our case. We summarize, in our context, their result.

Lemma 2.1 (Normalized Form)**.** *Any diagonal form as in* (1) *with coefficients in O_K is equivalent to a form of type*

$$\mathscr{F}_0 + \pi\mathscr{F}_1 + \cdots + \pi^{d-1}\mathscr{F}_{d-1} \tag{4}$$

where each $\mathscr{F}_j$ is a diagonal form with coefficients in O_K in m_j variables, and if x_i is a variable in $\mathscr{F}_j$ then its coefficient is not divisible by π. Furthermore, we have

$$N = \sum_{i=0}^{d-1} m_i \quad \text{and} \quad m_0 + \cdots + m_j \geq (j+1) \cdot \left(\frac{N}{d}\right) \tag{5}$$

for $j = 0, \ldots, d-2$. A form satisfying these properties is said to be normalized, and if a normalized form has non-trivial zeros in O_K, so does any equivalent form.

Now we define the concept of **contraction**, introduced by Davenport and Lewis [3]. Consider a form as in (4). Let $x_1, \ldots, x_t$ be variables of this form at levels less than j. If we can find $b_1, \ldots, b_t \in O_K$ such that

$$a_1 b_1^d + \cdots + a_t b_t^d = \pi^k \cdot m$$

with $k \geq j$ and $m \not\equiv 0 \bmod \pi$, then setting $x_i = b_i T$ for all $1 \leq i \leq t$ we obtain a new variable T at level k having coefficient $\pi^k \cdot m$. This process is called a *contraction of variables to a new variable at level k.* The method of contracting variables becomes a powerful tool when associated with the following versions of Hensel's Lemma (see [2] and [1], respectively).

Lemma 2.2 (Hensel - Unramified Case)**.** *Let K be the unramified quadratic extension of $\mathbb{Q}_2$. Suppose $\mathscr{F}$ is an additive form with coefficients in O_K and degree d. Let l be defined by $d = 2^l \cdot m$ with odd m. Let x be a variable in $\mathscr{F}$ at level j. Suppose x can be used in a contraction of variables (or in a series of contractions) which produces a new variable at level k. If*

$$k \geq \begin{cases} j + l + 2 & \text{for } l \geq 1 \\ j + l + 1 & \text{for } l = 0 \end{cases}$$

then $\mathscr{F}$ admits a non-trivial zero.

Lemma 2.3 (Hensel - Ramified Case)**.** *Let K be any ramified quadratic extension of $\mathbb{Q}_2$. Suppose $\mathscr{F}$ is an additive form with coefficients in O_K and degree d. Let l be defined by $d = 2^l \cdot m$ with odd m. Let x be a variable in $\mathscr{F}$ at level j. Suppose x can be used in a contraction of variables (or in a series of contractions) which produces a new variable at level $k \geq j + 2l + 3$. Then $\mathscr{F}$ admits a non-trivial zero.*

Definition 1. *A variable at level 0, or a variable at a higher level that was obtained by contractions containing variables at level 0, will be called a* ***primary*** *variable. Furthermore, a variable is* ***(i)-primary*** *if it is primary and has level i. The other variables will be called* ***(i)-secondary*** *according to their levels. We denote by p_i the number of primary variables at level i and by s_i the number of secondary variables at level i. Unlike the numbers m_i defined above, the p_i and s_i will not be fixed numbers, but will update as we make contractions. For example, if we have $p_3 = 5$ and $p_4 = 0$, and contract two (3)-primaries to make a (4)-primary, we will then have $p_3 = 3$ and $p_4 = 1$. However, the numbers m_i always represent the numbers of variables present before any contractions are made and do not change during the course of the proof.*

Remark 1. *Using the aforementioned versions of Hensel's Lemma, if K is the unramified quadratic extension of $\mathbb{Q}_2$, in order to obtain a non-trivial zero for $\mathscr{F}$, it is sufficient to construct an $(l + 2)$-primary for the cases where $l \geq 1$ or an $(l+1)$-primary for the cases where $l = 0$. Similarly, if K is any ramified quadratic extension of $\mathbb{Q}_2$, it is sufficient to construct an $(2l + 3)$-primary.*

3. Combinatorial Lemmas: Unramified Extension

The following lemmas will be our basic tools when dealing with contractions on the case of the unramified quadratic extension of $\mathbb{Q}_2$.

Lemma 3.1. (a) *If we have four variables at level k, then we can contract exactly two of them to a new variable at level at least $k + 1$.*

(b) *If we have three variables at level k, then we can contract three (two, if the zeroterms are not all distinct from each other) variables to a new variable at level at least $k + 1$.*

(c) *If we have three variables at level k with the same zeroterm, then we can contract exactly two variables to a new variable at level $k + 1$.*

PROOF. The first two statements are easy to prove, they follow directly from the additive structure of $O_K/(2) \cong \mathbb{F}_4$. We prove the last statement. Without loss of generality, we assume $k = 0$ and that the zeroterm is $a_0 = 1$. Since we are interested in contracting variables to level 1, we can work modulo 2^2. Consider three coefficients (modulo 2^2)

$$\alpha_1 = 1 + 2\beta_1, \quad \alpha_2 = 1 + 2\beta_2, \quad \alpha_3 = 1 + 2\beta_3$$

where β_1, β_2 and β_3 are representatives of $O_K/(2) \cong \mathbb{F}_4$.
If $\alpha_1 + \alpha_2 \equiv 0 \bmod 2^2$, then we have $(1 + \beta_1 + \beta_2) \equiv 0 \bmod 2$. It is easy to see that this occurs just in the four following cases.
Case 1: $\beta_1 = 1, \beta_2 = 0$ (or $\beta_1 = 0, \beta_2 = 1$).
If $\beta_3 = 0$, then $\alpha_2 + \alpha_3$ has level 1, and setting $x_2 = x_3 = T$ gives a new variable at level 1. If $\beta_3 = 1$, then $\alpha_1 + \alpha_3$ has level 1, and setting $x_1 = x_3 = T$ works. Finally, if $\beta_3 \notin \{0, 1\}$ we have $\alpha_1 + \alpha_3$ and $\alpha_2 + \alpha_3$ both at level 1, and either contraction works.

Case 2: $\beta_1 = a \notin \{0, 1\}, \beta_2 = a + 1$ (or $\beta_2 = a \notin \{0, 1\}, \beta_1 = a + 1$).
If $\beta_3 = a$, then $\beta_1 + \beta_3$ has level 1, and we can use the contraction $x_1 = x_3 = T$. If $\beta_3 = a + 1$, then $\beta_2 + \beta_3$ has level 1, and we can use the contraction $x_2 = x_3 = T$. If $\beta_3 \in \{0, 1\}$, then $\alpha_1 + \alpha_3$ and $\alpha_2 + \alpha_3$ both have level 1, and either contraction works. □

Lemma 3.2. *If we have $2M - 5$ secondary variables at level k with $M \geq 6$, then we can get $M - 5$ secondary variables at level $k + 1$ and are left with five variables at level k.*

PROOF. Since $M \geq 6$ we have $2M - 5 \geq 7$. By the pigeonhole principle, given seven variables at level k, at least three of them have the same zeroterm (remember that the residue field in this case is $\mathbb{F}_4$). So, from each seven variables we can contract two of them to a new variable at level $k + 1$ by Lemma 3.1(c). Repeating this procedure we have (see Definition 1)

$$s_{k+1} \geq \left\lfloor \frac{s_k - 7}{2} \right\rfloor + 1 = \left\lfloor \frac{s_k - 5}{2} \right\rfloor \geq \left\lfloor \frac{(2M - 5) - 5}{2} \right\rfloor = M - 5, \qquad (6)$$

as desired. □

Corollary 3.3. *Let $M \geq 6$ and recall the notation of Definition 1.*

(a) *If $s_k \geq 2^{t+1}M - 5$, then we can obtain by contraction $M - 5$ new $(k+t+1)$-secondaries, leaving five variables at all levels from k to $k + t$.*

(b) *If* $s_k + \cdots + s_{k+t} \geq 2^{t+1}M - 5$, *we can obtain by contraction* $M-5$ *new* $(k+t+1)$*-secondaries.*

Proof. First we prove part (a). Let us denote by s_j^* the number of $(k+j)$-secondaries we get after contracting secondary variables at lower levels. It follows from (6) that

$$s_{k+1}^* = \left\lfloor \frac{s_k - 5}{2} \right\rfloor + s_{k+1} \geq 2^t M - 5, \tag{7}$$

and more generally, for $j = 1, \ldots, t$

$$s_{k+j+1}^* = \left\lfloor \frac{s_{k+j}^* - 5}{2} \right\rfloor + s_{k+j+1}. \tag{8}$$

What is happening here is precisely the update on the secondaries we mentioned in Definition 1. A simple inductive argument applied to (7) gives

$$s_{k+j}^* \geq 2^{t+1-j}M - 5 \quad \text{and} \quad s_{k+t+1} \geq M - 5.$$

Since we are using Lemma 3.2 recursively, we leave 5 variables at every lower level, completing the proof of part (a).

The proof of part (b) will be by induction. Observe that if $s_{k+t} \geq 2M-5$, the result follows directly from Lemma 3.2. Let us assume as our inductive hypothesis that if $s_{k+t-r} + \cdots + s_{k+t} \geq 2^{r+1}M - 5$ then we can obtain $M-5$ new $(k+t+1)$-secondaries. Now assume that

$$s_{k+t-r-1} + s_{k+t-r} + \cdots + s_{k+t} \geq 2^{r+2}M - 5.$$

We may assume $s_{k+t-r} + \cdots + s_{k+t} \leq 2^{r+1}M - 6$, otherwise we are done, by the inductive hypothesis. Hence

$$s_{k+t-r-1} \geq 2^{r+2}M - 5 - (2^{r+1}M - 6) = 2^{r+1}M + 1 \geq 2^{r+1} \cdot 6 + 1. \tag{9}$$

From (8) we have

$$s_{k+t-r}^* = \left\lfloor \frac{s_{k+t-r-1} - 5}{2} \right\rfloor + s_{k+t-r} = \left\lfloor \frac{s_{k+t-r-1} + 2s_{k+t-r} - 5}{2} \right\rfloor.$$

An inductive argument gives

$$\begin{aligned} s_{k+t}^* &= \left\lfloor \frac{s_{k+t-r-1} + 2s_{k+t-r} + \cdots + 2^{r+1}s_{k+t} - 5(\sum_{i=0}^r 2^i)}{2^{r+1}} \right\rfloor \\ &\geq \left\lfloor \frac{s_{k+t-r-1} + s_{k+t-r} + \cdots + s_{k+t} - 5(2^{r+1} - 1)}{2^{r+1}} \right\rfloor, \end{aligned}$$

and it follows from (9) that the numerator of the fraction is always positive. Hence

$$s^*_{k+t} \geq \left\lfloor \frac{2^{r+2}M - 5 - 5(2^{r+1}-1)}{2^{r+1}} \right\rfloor = 2M - 5,$$

and the result follows from Lemma 3.2. □

Lemma 3.4. *If we have $2M-1$ primary variables at level k with $M \geq 2$, then we can use contractions to form $M-1$ primary variables at levels at least $k+1$, leaving at most one variable at level k.*

Proof. Indeed, if $M = 2$, then we have three primaries at level k and the result follows from Lemma 3.1(b). If $M > 2$, then from each four variables at level k we can contract two of them to a new variable at level at least $k+1$ (Lemma 3.1(a)). After these contractions have been performed, we still have three variables unused and by Lemma 3.1(b) we can get one extra primary variable. So the number of primaries at levels at least $k+1$ is

$$\left\lfloor \frac{p_k - 3}{2} \right\rfloor + 1 = \left\lfloor \frac{p_k - 1}{2} \right\rfloor \geq \left\lfloor \frac{(2M-1)-1}{2} \right\rfloor = M - 1. \tag{10}$$

□

Corollary 3.5. *Let $M \geq 2$ and recall the notation in Definition 1.*

(a) *If we have $p_k \geq 2^{t+1}M - 1$ with $M \geq 2$, then we can use contractions to form $M-1$ primary variables at levels at least $k+t+1$, leaving at most one variable at each level from k to $k+t$.*

(b) *If $p_k + \cdots + p_{k+t} \geq 2^{t+1}M - 1$ then we can use contractions to form $M-1$ primary variables at levels at least $k+t+1$.*

Proof. For part (a), observe that by Lemma 3.4, we will have $2^tM - 1$ primaries at levels at least $k+1$. If some of these variables are at levels at least $k+2$, we would need to apply Lemma 3.4 to fewer $(k+1)$-primaries to obtain $2^{t-1}M - 1$ primaries at levels at least $k+2$. In this sense, the worst case to consider would be all variables obtained by an application of Lemma 3.4 landing at the next immediate level. With this understanding, we apply Lemma 3.4 recursively to the $2^tM - 1$ primaries at level $k+1$ to obtain the result.

For part (b), with the same understanding above, the proof of this result follows the same lines as the proof of Corollary 3.3(b), but now using Lemma 3.4 and (10) instead of Lemma 3.2 and (6). □

Lemma 3.6. *If $p_k + \cdots + p_{k+t} \geq 2^{t+1}M$, with $M \geq 1$, and $s_{k+j} \geq 1$ for $j = 0, 1, \ldots, t$, then we can get M primary variables at levels at least $k+t+1$.*

Proof. Here we will follow the same understanding as in the proof of Corollary 3.5 and consider the jumps on the contracted variables being of size exactly one. Furthermore, since any contraction involving the secondary variables must necessarily include a primary variable, we can consider, for contractions sake, that we have at every level an extra primary.

Thus we can now go back to (10) and consider that the number of primary variables obtained from the $p_k + 1$ (k)-primaries is at least

$$\left\lfloor \frac{(p_k+1)-3}{2} \right\rfloor + 1 = \left\lfloor \frac{p_k}{2} \right\rfloor . \tag{11}$$

In particular, if $p_k \geq 2M$, then we can get M primaries at levels at least $k+1$. With the same understanding presented in the proof of Corollary 3.5(b), we can apply the same inductive argument to conclude the proof. □

4. Combinatorial Lemmas: Ramified Extensions

The following lemmas will be our basic tools when dealing with contractions on the cases of the ramified quadratic extensions of $\mathbb{Q}_2$.

Lemma 4.1. (a) *If we have two variables at level k, then they contract to a new variable at level at least $k+1$.*

(b) *If we have three variables at level k, then we can contract two of them to a new variable at level at least $k+2$.*

(c) *If we have five variables at level k, then we can contract exactly two variables to a new variable at level $k+2$.*

Proof. The first statement follows directly from the additive structure of $O_K/(\pi) \cong \mathbb{F}_2$. To prove parts (2) and (3) we assume $k = 0$ and work modulo π^3. Suppose we have five variables at level 0 with coefficients $a_i = 1 + \alpha_i \pi + \beta_i \pi^2$, $i = 1, \ldots, 5$. Each set of three coefficients, say a_1, a_2, a_3, has two elements satisfying $\alpha_i = \alpha_j$. Contracting the corresponding variables by setting $x_i = x_j = T$, we get a new variable at level at least 2 (remember that $2 \equiv \pi^2 \bmod \pi^3$). Similarly, two of the five coefficients satisfy $\alpha_i = \alpha_j$ and $\beta_i = \beta_j$. Setting $x_i = x_j = T$ gives us a new variable at level 2. □

Lemma 4.2. *If we have $2M-3$ variables at level k, with $M \geq 4$, then we can choose between the following two options.*

a) *We can use contractions to form $M-3$ variables at level $k+2$ and another variable at level at least $k+2$.*

b) *We can use contractions to form $M-3$ variables at level $k+2$ and leave three variables at level k.*

Proof. Without loss of generality we assume the variables being secondary. By Lemma 4.1(c), from each set with five variables we can find two that contract to a new variable at level $k+2$. Applying this successively we get (see Definition 1)

$$s_{k+2} \;\geq\; \left\lfloor \frac{s_k-5}{2} \right\rfloor + 1 \;=\; \left\lfloor \frac{s_k-3}{2} \right\rfloor \;\geq\; \left\lfloor \frac{(2M-3)-3}{2} \right\rfloor = M-3. \qquad (12)$$

In this process we are left with three variables at level k not contracted. Then we can either leave them at level k or apply Lemma 4.1(b) and get another variable at level at least $k+2$. □

Corollary 4.3. *Let $M \geq 4$ and recall the notation of Definition 1.*

(a) *If $s_k \;\geq\; 2^{t+1}M-3$, then we can obtain by contraction $M-3$ new variables at level $k+2(t+1)$, and another extra variable a level at least $k+2(t+1)$.*

(b) *If $s_k+s_{k+2}+\cdots+s_{k+2t} \geq 2^{t+1}M-3$, we can obtain by contraction $M-3$ new variables at level $k+2(t+1)$, leaving three variables at level $k+2t$.*

Proof. Without loss of generality we assume the variables being secondary. The proof follows exactly the same lines as the proof of Corollary 3.3. The only difference is that now we are applying recursively Lemma 4.2, having (see (12))

$$s^*_{k+2} \;=\; \left\lfloor \frac{s_k-3}{2} \right\rfloor + s_{k+2} \;\geq 2^t M - 3,$$

and for $j=1,\ldots,t+1$,

$$s^*_{k+2j} \;=\; \left\lfloor \frac{s^*_{k+2(j-1)}-3}{2} \right\rfloor + s_{k+2j}.$$

□

Lemma 4.4. *Suppose we have two variables at level k, one of them being primary, and assume that at each level $k+j$ $(j=1,\ldots,s)$, we have one variable (not necessarily primary). Then we can obtain a primary variable at level at least $k+s+1$.*

Proof. Applying Lemma 4.1(a), we contract the two variables at level k and get a new primary variable at level $k+r$ for some $r \in \mathbb{N}$. If $r \geq s+1$ we are done. If not, just repeat the process. □

Lemma 4.5. *Suppose we have two variables at level k (one of them being primary) and two variables at level $k+1$ (one of them being primary). Then we can obtain a primary variable at level at least $k+3$.*

Proof. We apply Lemma 4.1(a) and contract the two variables at level k to a new primary variable at level at least $k+1$. If the new variable is at level $k+3$ or higher we have nothing to do. If it is at level $k+1$ we have three variables at this level, two of them being primary, and Lemma 4.1(b) gives us a new primary at level at least $k+3$. Finally, if it is at level $k+2$, we are done by Lemma 4.4. □

Lemma 4.6. *Suppose we have four primary variables at level k, a fifth primary at level at least k, three secondaries at level k and one secondary at level $k+2$. Then we can obtain two primary variables at levels at least $k+3$.*

Proof. We divide the proof according to the level of the fifth primary, which we denote by T_0.

Case 1: The primary T_0 is at level at least $k+3$.
In this case, we apply Lemma 4.1(b) to the (k)-primaries to get a new primary at level $k+2$ (if it has higher level then we are done) which contracts with the $(k+2)$-secondary to a second primary at level at least $k+3$.

Case 2: The primary T_0 is at level $k+2$.
We contract this variable with the $(k+2)$-secondary and get a primary at level at least $k+3$. Then we apply Lemma 4.1(b) to a set with four (k)-primaries and one (k)-secondary to obtain two primaries at level $k+2$ (if one has higher level we are done). Then we contract these two primaries and get the result.

Case 3: The primary T_0 is at level $k+1$.
By Lemma 4.1(b) we can contract two of the four (k)-primaries to a primary at level at least $k+2$, and this primary together with the $(k+2)$-secondary will give a primary at level at least $k+3$. Pairing up the two remaining (k)-primaries with two (k)-secondaries we obtain two primaries at level at least $k+1$. If one of these two primaries is at level at least $k+3$, we are done. If one of them is a $(k+1)$-primary and the other is a $(k+2)$-primary, we can include T_0 and apply Lemma 4.4 to obtain a primary at level at least $k+3$. If they are both $(k+1)$-primary, together with T_0, we can obtain a primary at level at least $k+3$, according to Lemma 4.1(b).

Case 4: The primary T_0 is at level k.

We now have five (k)-primaries. By Lemma 4.1(c) we can contract two of them to obtain a $(k+2)$-primary, which together with the $(k+2)$-secondary gives a primary at level at least $k+3$. Pairing up the three remaining (k)-primaries with three (k)-secondaries, we obtain three primaries at levels at least $k+1$ and at most $k+2$, otherwise we conclude the proof. If we have, among these three primaries, either two $(k+2)$-primaries or three $(k+1)$-primaries, the result will follow from Lemma 4.1, parts (a) or (b) respectively. Thus we may assume we have two $(k+1)$-primaries and one $(k+2)$-primary. Now the result follows from Lemma 4.4. This concludes the proof. □

Lemma 4.7. *Suppose we have three primary variables at level k, a fourth primary at level at least k, one secondary at level k and two secondaries at level $k+2$. Then we can obtain two primary variables at levels at least $k+3$.*

Proof. Our goal is to produce two primaries at levels at least $k+2$, for in this case we can pair them up with the two $(k+2)$-secondaries to produce the two primary variables at levels at least $k+3$. We will divide the proof according to the level of the fourth primary, which we denote by T_0.
Case 1: The primary T_0 is at level at least $k+2$.
We apply Lemma 4.1(b) to the (k)-primaries to get a second primary at level at least $k+2$.

Case 2: The primary T_0 is at level $k+1$.
We apply Lemma 4.1(b) to the (k)-primaries, contracting two of them to a primary at level at least $k+2$. Then we apply Lemma 4.4 to the remaining (k)-primary, one (k)-secondary, and T_0, and then we obtain a second primary at level at least $k+2$.

Case 3: The primary T_0 is at level k.
We apply Lemma 4.1(c) and then Lemma 4.1(b) to a set with four (k)-primaries (including T_0) and one (k)-secondary, and construct two primaries at levels at least $k+2$, concluding the proof. □

Lemma 4.8. *Suppose one of these situations holds*

(a) *there exist two $(k+1)$-primaries;*

(b) *there exist two (k)-primaries and one variable (not necessarily primary) at either level k or $k+1$;*

(c) *there exist one (k)-primary, one $(k+1)$-primary and one variable (not necessarily primary) at either level k or $k+1$.*

Then we can obtain a primary variable at level at least $k+2$.

PROOF. If we have two primaries at level $k+1$ we contract them and are done. Assume that the two primaries are at level k. If the third variable is also at level k then we are done by Lemma 4.1(b), and if the third variable is at level $k+1$ then we are done by Lemma 4.4. Finally, if we have one primary variable at level k and the other at level $k+1$, we use Lemma 4.4 if the third variable is at level k and Lemma 4.1(a) if it is at level $k+1$. □

Remark 2. *The next definition will be critical in our proof for ramified extensions of $\mathbb{Q}_2$. It follows from Remark 1 that, in order to prove Theorem 1, it is sufficient to use a variable from level 0 in a series of contractions to produce a primary variable at level $2\tau+3$. However, this may not always be possible. During the course of the proof we will find situations where we are unable to make such contractions, so as an alternative we will use a variable from level 1 in a series of contractions to produce a variable at level $2\tau+4$. By Lemma 2.3, this also guarantees that $\mathscr{F}$ has a non-trivial zero.*

Definition 2. *We will use the term* ***B-primary*** *to refer to any variable at level 1, or any variable derived from such a variable through contractions, and refer to a B-primary variable at level k as a* $(\mathbf{k})$-***B-primary*** *variable. Then the usual primary variables will be called* ***A-primary***. *Naturally, a variable can be of types A and B simultaneously. If it is not explicitly said that a variable is A-primary or B-primary then this variable is secondary.*

Lemma 4.9. *Suppose we have six A-primaries at level k, three B-primaries at level $k+1$, another B-primary at level at least $k+1$ and one secondary at level $k+5$. Then one of the following possibilities occurs:*

(a) *we can make an A-primary at level at least $k+5$.*

(b) *we can make a B-primary at level at least $k+6$.*

PROOF. By Lemma 4.1(b) we can contract two B-primaries at level $k+1$ to obtain a new B-primary S_0 at level at least $k+3$. In fact we can assume S_0 to be at level at most $k+4$, otherwise we would either have part (b) directly or contracting S_0 with the $(k+5)$-secondary would give part (b), according to Lemma 4.1(a). Observe that we still have one B-primary S_1 at level $k+1$. Now we can use Lemma 4.2 to contract two (k)-A-primaries to an A-primary T_0 at level $k+2$. By Lemma 4.1(b) we can contract two of the remaining four (k)-A-primaries to a new A-primary T_1 at level at least $k+2$ and at most $k+4$, otherwise we obtain part (a). After all these contractions are performed we have:
S_0 at level either $k+3$ or $k+4$, S_1 at level $k+1$, another B-primary at level between $k+1$ and $k+4$ (by the same reasoning we applied to S_0), T_0 at level $k+2$,

T_1 at level between $k+2$ and $k+4$, two (k)-A-primaries, and a $(k+5)$-secondary.

Case 1: The B-primary S_0 is at level $k+4$.
If T_1 is at level $k+4$, together with S_0, then we get an A-primary at level at least $k+5$, by Lemma 4.1(a), confirming part (a). If T_1 is at level $k+3$, we would have two (k)-A-primaries and also variables at levels $k+1$, $k+2$, $k+3$ and $k+4$ (the variables S_1, T_0, T_1 and S_0 respectively). By Lemma 4.4, these variables suffice to obtain part (a). So assume T_1 is at level $k+2$. Now contract the two (k)-A-primaries to get an A-primary T_2 at level at least $k+1$ (see Lemma 4.1(a)) and at most $k+4$ (otherwise we get part (a)). If T_2 is at level $k+4$ with S_0 then we get (a) (see Lemma 4.1(a)). If T_2 is at level $k+3$ we would have, together with T_0 and T_1 at level $k+2$ and S_0 at level $k+4$, part (a) according to Lemma 4.4. If T_2 is at level $k+2$, together with T_0 and T_1 also at level $k+2$, they would give an A-primary at level at least $k+4$, by Lemma 4.1(b). But this new variable is either at level $k+5$, or together with S_0 could be contracted to an A-primary at level at least $k+5$, confirming part (a). Finally assume T_2 is at level $k+1$. Now we apply Lemma 4.5 to the variables S_1, T_2 at level $k+1$ and T_0, T_1 at level $k+2$ and obtain an A-primary at level at least $k+4$. This gives us part (a) since either this new A-primary is already at level at least $k+5$ or together with S_0 it can be contracted to an A-primary at level at least $k+5$, concluding the proof of this case.

Case 2: The B-primary S_0 is at level $k+3$.
If T_1 is at level $k+4$ we can apply Lemma 4.4 to the two (k)-A-primaries, and to S_1, T_0, S_0 and T_1 to obtain an A-primary at level at least $k+5$, confirming part (a). Assume T_1 is at level $k+3$. We can again apply Lemma 4.4, now to the two (k)-A-primaries, S_1, and T_0, to obtain an A-primary T_2 at level at least $k+3$ and at most $k+4$ (otherwise we get part (a)). If T_2 is at level k+4 we can obtain part (a) by applying Lemma 4.4 to S_0, T_1 at level $k+3$, and T_2 at level $k+4$. Thus assume T_2 is at level $k+3$. Now we have T_1, T_2 and S_0 at level $k+3$, and this is enough to guarantee part (a), by Lemma 4.1(b). Now assume T_1 is at level $k+2$. Besides S_0 there is an extra B-primary S_2 at level at least $k+1$. We can assume the level of S_2 is at most $k+3$. Indeed, if it is higher than $k+5$ we get part (b); if it is $k+5$ we contract S_2 with the $(k+5)$-secondary obtaining part (b); and if it is $k+4$ we get part (a) or part (b), by Case 1 above. If S_2 is at level $k+3$, we can apply Lemma 4.5 to T_0, T_1, S_0 and S_2 to obtain a primary variable at level at least $k+5$. If this primary is an A-primary we get part (a). If this primary is a B-primary then either its level is higher than $k+5$ giving us part (b) directly or it can be contracted together with the $(k+5)$-secondary and

again we obtain part (b). If S_2 is at level $k+2$, we can apply Lemma 4.1(b) to S_2, T_0, and T_1 to obtain an A-primary T_2 at level $k+4$ (otherwise we get part (a)), leaving one primary T^* at level $k+2$. By Lemma 4.4, we can contract the two (k)-A-primaries, S_1, T^*, S_0, and T_2 to obtain part (a). Finally assume S_2 is at level $k+1$. Then we can contract the two (k)-primaries, S_1, and S_2 to obtain a primary S^* at level at least $k+3$ (Lemma 4.5). Suppose S^* is at level at least $k+5$. If S^* is an A-primary, then we have part (a). If S^* is a B-primary it can be contracted with the $(k+5)$-secondary giving us part (b). Assume then that S^* is at level at most $k+4$. Now we have T_0, T_1 at level $k+2$, S_0 at level $k+3$ and S^* at level $k+3$ or $k+4$. Depending on the level of S^* we can apply either Lemma 4.4 or Lemma 4.5 to obtain a primary variable U at level at least $k+5$. If U is A-primary we get part (a), and if U is B-primary then either its level is higher than $k+5$ or together with the $(k+5)$-secondary we get part (b). This completes the proof of the lemma. □

Lemma 4.10. *Suppose we have five A-primaries at level k, three B-primaries at level $k+1$, one A-primary at level at least $k+1$, one B-primary at level at least $k+1$ and one secondary at level $k+5$. Then one of the following possibilities occurs.*

(a) *We can make an A-primary at level at least $k+5$.*

(b) *We can make a B-primary at level at least $k+6$.*

Proof. To prove Lemma 4.10 we proceed as in the proof of Lemma 4.9. The only difference is that the points where we had to contract the last two k-primaries to get another one at level at least $k+1$ no longer exist since we are assuming that we have an extra primary at level at least $k+1$. All the other constructions we did in the proof of Lemma 4.9 can still be performed here. □

5. Proof of Theorem 1

From this point on we consider the form $\mathscr{F}$ to be normalized, with coefficients in O_K, degree $d = 2^l \cdot m$ with $m \geq 3$ odd, and in $N = d^2+1$ variables. According to Lemma 2.1 we may assume that

$$\mathscr{F}(x_1, \ldots, x_N) = \mathscr{F}_0 + \pi\mathscr{F}_1 + \cdots + \pi^{d-1}\mathscr{F}_{d-1}, \tag{13}$$

with all the properties stated in the lemma. In particular, we assume (see (5)), for $j = 0, \ldots, d-2$,

$$\sum_{i=0}^{j} m_i \geq (j+1)\frac{N}{d} \geq 3(j+1)\cdot 2^l + 1 \quad \text{and} \quad \sum_{i=0}^{d-1} m_i = d^2 + 1 \geq 9 \cdot 2^{2l} + 1. \tag{14}$$

The proof will be conducted by analyzing separately the cases where K is the unramified extension of $\mathbb{Q}_2$ or one of the ramified extensions. For a better understanding, we divide the proof in each case into a few propositions, according to the value of l in the degree $d = 2^l \cdot m$, with $m \geq 3$. But first we consider the easier case of d odd.

Proposition 5.1. *Let K be a quadratic extension of $\mathbb{Q}_2$. Any normalized form $\mathscr{F}$ of odd degree $d \geq 3$, in $N = d^2 + 1$ variables has non-trivial zeros over K.*

Proof. It follows from (14) that

$$m_0 \geq 4 \quad \text{and} \quad m_0 + m_1 \geq 7.$$

If K is the unramified extension, it follows from Remark 1 that it suffices to obtain a primary variable at level at least 1. Since $m_0 \geq 4$ the result is easily attained by an application of Lemma 3.1.

Suppose K is a ramified extension. Now we need to obtain a primary variable at level at least 3. If $m_0 \geq 5$ $(= 2\cdot 4 - 3)$, we can use Lemma 4.2(a) to obtain two (2)-primaries (otherwise we are done). Now the result follows from a simple application of Lemma 4.1(a). Thus we can assume

$$m_0 = 4 \quad \text{and} \quad m_1 \geq 3.$$

By Lemma 4.1(b) we can obtain a primary variable at level 2 (if it is at a higher level we are done). Then contracting the two remaining primaries at level 0 we get the result by Lemma 4.4. □

From now on we will analyse separately the cases of the quadratic extension of $\mathbb{Q}_2$ being unramified or ramified.

Proof of theorem 1 for the unramified quadratic extension of $\mathbb{Q}_2$.

Our goal is to obtain a primary variable at level at least $l+2$, for this suffices to prove Theorem 1 when K is unramified, according to Remark 1.

Our starting point is the observation that if $m_0 \geq 2^{l+2} \cdot 2 - 1$ we can obtain a primary at level at least $l+2$ as a direct application of Corollary 3.5, completing the proof of Theorem 1. Therefore we may assume from now onwards (see (14))

$$3 \cdot 2^l + 1 \ \leq m_0 \ \leq 2^{l+2} \cdot 2 - 2 \quad \text{and} \quad m_1 + \cdots + m_j \geq 3(j+1) \cdot 2^l + 1 - m_0 \quad (15)$$

for $j = 1, \ldots, l+1$.

Remark 3 (General Strategy)**.** *The idea that will be extensively used here (and also in the ramified case) is the following: once we have a bound on m_0, an application of Corollary 3.5 (Corollary 4.3 for the ramified cases) will give us primaries at higher levels. Then we start a series of considerations about the possibilities of the existence of secondaries at higher levels and their consequences in obtaining primary variables at even higher levels. But after these initial cases are considered, the aftermath is that we are left with bounds for some values of the m_j. In general we will end up in a situation having, for example, $m_0 \leq K_0$, $m_t \leq K_1$ and $m_{t+1} \leq K_2$. Hence when we write an inequality (see (15)) of the type*

$$m_1 + \cdots + m_{t-1} \geq M \qquad (16)$$

we mean that the value M was obtained as $M = 3(t+2)2^l + 1 - K_0 - K_1 - K_2$.

There are many cases to be considered according to the value of m_0, and we start with the next Lemma.

Lemma 5.2. *Let K be the unramified quadratic extension of $\mathbb{Q}_2$ and $\mathscr{F}$ be a normalized form of degree $d = 2^l \cdot m$ with $m \geq 3$ odd and $l \geq 2$, in $N = d^2 + 1$ variables. If $m_0 \geq 2^l \cdot 4 + 1$ then $\mathscr{F}$ has non-trivial zeros over K.*

Proof. We are going to create an $(l+2)$-primary. We divide this proof in a few cases according to the value of m_0. By (15) we can assume $m_0 \leq 2^l \cdot 8 - 2$.

Case 1: $2^l \cdot 6 - 1 \leq m_0 \leq 2^l \cdot 8 - 2$.
By Corollary 3.5, since $p_0 \geq 2^{l+1} \cdot 3 - 1$ we obtain two primaries at levels at least $l+1$. If one of them is at level at least $l+2$, we conclude the proof, so we may assume we have $p_{l+1} \geq 2$. If $m_{l+1} \geq 1$ we are done by Lemma 3.6, so we assume $m_{l+1} = 0$. In this case (14) gives us (see Remark 3)

$$m_1 + \cdots + m_l \geq ((l+2)2^l \cdot 3 + 1) - m_0 - m_{l+1} \geq 2^l \cdot 6 - 5,$$

since $l \geq 2$. Then by Corollary 3.3 we get $s_{l+1} \geq 1$ and we are done.

Case 2: $2^l \cdot 5 - 1 \leq m_0 \leq 2^l \cdot 6 - 2$.
By Corollary 3.5 we obtain $p_l \geq 4$. If $m_l \geq 1$, Lemma 3.6 gives us $p_{l+1} \geq 2$ and then we proceed as we did in the above case, for the number of secondaries is now larger and we used only one (l)-secondary. If $m_l = 0$ then (see (14))

$$m_1 + \cdots + m_{l-1} \geq ((l+1)2^l \cdot 3 + 1) - m_0 - m_l \geq 2^{l-1} \cdot 10 - 5.$$

By Corollary 3.3 we obtain $s_l \geq 5$ and we can proceed as in case $m_l \geq 1$.

Case 3: $2^{l-1} \cdot 9 - 1 \leq m_0 \leq 2^l \cdot 5 - 2$.
By Corollary 3.5 we get $p_{l-1} \geq 8$. Suppose $m_{l-1} \geq 1$. Then Lemma 3.6 gives us $p_l \geq 4$. If $m_l \cdot m_{l+1} \neq 0$ then Lemma 3.6 gives us the result. Note that assuming $m_{l+1} = 0$ we get (see (14))

$$m_1 + \cdots + m_l - 2 \geq ((l+2)2^l \cdot 3 + 1) - m_0 - m_{l+1} - 2 \geq 2^l \cdot 8 - 5$$

(here we are assuming $l \geq 2$ and excluding two secondaries that we already used to obtain $p_{l+1} \geq 2$) and by Corollary 3.3 we get $s_{l+1} \geq 3$. Similarly, assuming $m_l = 0$ we would get

$$m_1 + \cdots + m_{l-1} - 1 \geq ((l+1)2^l \cdot 3 + 1) - m_0 - m_l - 1 \geq 2^{l-1} \cdot 10 - 5$$

and Corollary 3.3 gives us $s_l \geq 5$. Naturally, if it were $m_l = 0 = m_{l+1}$ the same argument would give us $s_l \cdot s_{l+1} \neq 0$ and we can proceed as we did when $m_l \cdot m_{l+1} \neq 0$. If $l = 2$, then this completes the proof of the lemma, since $2^{2-1} \cdot 9 - 1 = 2^2 \cdot 4 + 1$ and $m_1 \neq 0$.

Finally, assume that $l \geq 3$ and $m_{l-1} = 0$. Then we would get

$$m_1 + \cdots + m_{l-2} \geq (3l \cdot 2^l + 1) - m_0 - m_{l-1} \geq 2^{l-2} \cdot 16 - 5$$

and by Corollary 3.3 we may use at most $2^{l-2} \cdot 6 - 5$ of these secondaries to get $s_{l-1} \geq 1$. We may now proceed as we did when $m_{l-1} \geq 1$, as even after excluding these $2^{l-2} \cdot 6 - 5$ secondaries, we are left with sufficient variables to guarantee $s_l \cdot s_{l+1} \neq 0$.

Case 4: $2^l \cdot 4 + 1 \leq m_0 \leq 2^{l-1} \cdot 9 - 2$.
From (14) we have $m_1 \geq 2^l \cdot 6 + 1 - m_0 \geq 2^{l-1} \cdot 3 + 3$. By Corollary 3.3 we can contract at most $2^{l-1} \cdot 3 - 2$ of these variables and get $s_{l-1} \geq 1$, with the

assurance that there are at least five secondaries left at every level from 1 to $l-2$. By Lemma 3.4 we have

$$p_1 + p_2 + \cdots + p_l + p_{l+1} \geq 2^{l-1} \cdot 4.$$

First we show that we can assume $p_l + p_{l+1} \geq 4$. Indeed, if we have $p_l + p_{l+1} = \delta$ with $0 \leq \delta \leq 3$, then

$$p_1 + \cdots + p_{l-1} \geq 2^{l-1} \cdot 4 - \delta \geq 2^{l-1} \cdot (4 - \delta)$$

and these variables together with the secondaries at levels $1, 2, \ldots, l-1$ can be contracted (Lemma 3.6) in order to get

$$p_l + p_{l+1} \geq 4 - \delta + \delta = 4.$$

We conclude that if it is $m_l \cdot m_{l+1} \neq 0$ we get the result by Lemma 3.6. But note that if we assume $m_{l+1} = 0$ we get (see (14))

$$m_1 + \cdots + m_l \geq ((l+2)2^l \cdot 3 + 1) - m_0 - m_{l+1} \geq 2^{l-1} \cdot 19 - 5$$

and even after excluding the $2^{l-1} \cdot 3 - 2$ secondaries we have already used we can apply Corollary 3.3 and obtain $s_{l+1} \geq 3$. Similarly, if it were $m_l = 0$ we would get

$$m_1 + \cdots + m_{l-1} \geq ((l+1)2^l \cdot 3 + 1) - m_0 - m_l \geq 2^{l-1} \cdot 13 - 5$$

and again we can use Corollary 3.3 ang get $s_l \geq 5$. Naturally if we have $m_l = 0 = m_{l+1}$ the same reasoning leads us to $s_l \cdot s_{l+1} \neq 0$.

□

Remark 4. *We note here that in the proof of Lemma 5.2, we did not really use the hypothesis that $N = d^2 + 1$. In reality, the proof only used the bounds on $m_0 + \cdots + m_j$ that we obtained for $0 \leq j \leq l+1$. In the remainder of the article, when we say that something is true "by the proof of Lemma 5.2," we mean that although we may not have $d^2 + 1$ variables, these bounds on $m_0 + \cdots + m_j$ still hold, and therefore the conclusion of this lemma holds.*

Proposition 5.3. *Let K be the unramified quadratic extension of $\mathbb{Q}_2$. Any normalized form $\mathscr{F}$ of degree $d = 2^l \cdot m$ with $m \geq 3$ odd and $l \geq 5$, in $N = d^2 + 1$ variables has non-trivial zeros over K.*

PROOF. We may assume $m_0 \leq 2^l \cdot 4$, according to Lemma 5.2, and also divide the proof in cases depending on the value of m_0.

Case 1: $2^{l-1} \cdot 7 + 1 \leq m_0 \leq 2^l \cdot 4$.
We apply Lemma 3.4 and contract the $2^{l-3} \cdot 28 + 1$ variables at level 0 to $2^{l-4} \cdot 28$ new primaries at higher levels. Then we have

$$p_1 + \cdots + p_{l+1} \geq 2^{l-4} \cdot 28. \tag{17}$$

We are excluding the case of primaries at levels at least $l+2$, otherwise we are done. We will analyze two subcases.

Subcase A**:** $\sum_{i\geq 2} p_i \geq 2^{l-4} \cdot 4$.
First we show that we can assume $\sum_{i\geq 2} p_i \geq 2^{l-5} \cdot 32$. Indeed, assuming otherwise we would get

$$\sum_{i\geq 2} p_i = 2^{l-4} \cdot 4 + \epsilon, \quad \text{with } 0 \leq \epsilon < 2^{l-5} \cdot 24.$$

Since $m_0 \leq 2^l \cdot 4$ we have $m_1 \geq 2^l \cdot 2 + 1 \geq 2^{l-2} \cdot 8 - 3$ (see (14)) and by Corollary 3.3 we get $s_j \geq 1$ for $j = 1, \ldots, l-1$. By (17) we have

$$p_1 \geq 2^{l-4} \cdot 24 - \epsilon \geq 2(2^{l-5} \cdot 24 - \epsilon)$$

and since $s_1 \geq 1$, Lemma 3.6 gives us $2^{l-5} \cdot 24 - \epsilon$ new primaries at levels at least 2. So we have

$$\sum_{i=2}^{l+1} p_i \geq 2^{l-4} \cdot 4 + \epsilon + 2^{l-5} \cdot 24 - \epsilon = 2^{l-5} \cdot 32.$$

Now let δ be the number of these primaries that are at levels higher than $l-3$. We can assume $\delta \geq 32$. Indeed, if that is not the case, we would have

$$\sum_{i=2}^{l-4} p_i \geq 2^{l-5} \cdot 32 - \delta \geq 2^{l-5} \cdot (32 - \delta)$$

and applying Lemma 3.6 to these variables and to the secondaries at levels $2, \ldots, l-4$ we obtain $32 - \delta$ new primaries at levels at least $l-3$. Since we already had δ primaries at these levels, we have $\sum_{i=l-3}^{l+1} p_i \geq 32$. Furthermore, we still have $s_j \geq 1$ for $j = l-3, l-2, l-1$. Assume $m_l \geq 7$. Then Lemma 3.2 gives us $s_l \geq 1$ and $s_{l+1} = 1$ and we apply Lemma 3.6 to the primaries and secondaries at levels at least $l-3$ and get the result. Now if $m_l \leq 6$, then we would have (see (14) and remember that $l \geq 5$)

$$m_1 + \cdots + m_{l-1} \geq (3l-1)2^l - 5 \geq 2^l \cdot 14 - 5.$$

Since we have already used $2^l \cdot 2 + 2$ of these variables (the (1)-secondaries contracted at the beginning of subcase A and possibly one more in our appeal to Lemma 3.6) we have

$$s_1 + \cdots + s_{l-1} \geq 2^l \cdot 12 - 7,$$

which suffices to guarantee $s_l \geq 7$ (Corollary 3.3) and we can proceed as above.

Subcase B**:** $\sum_{i\geq 2} p_i < 2^{l-4} \cdot 4.$

Here we have $p_1 \geq 2^{l-4} \cdot 24$. At this point we introduce the cyclic change of variables

$$\mathscr{F}^{(1)} = \frac{1}{2}\mathscr{F}(2x_1, \ldots, 2x_s, x_{s+1}, \ldots, x_N), \tag{18}$$

where $x_1, \ldots, x_s$ are the variables at level 0 that remained untouched. Since we are working over a field K, it is simple to see that any non-trivial zero of $\mathscr{F}^{(1)}$ gives a non-trivial zero for $\mathscr{F}$. Hence whenever it is convenient, we are going to apply this change of variables and proceed with the proof. We denote by $m_i^{(1)}$ the number of variables at level i in the form $\mathscr{F}^{(1)}$. Thus the variables of $\mathscr{F}^{(1)}$ satisfy

$$\begin{aligned} m_0^{(1)} \geq m_1 + p_1 \ &= \ m_0 + m_1 - (m_0 - p_1) \\ \geq 2^l \cdot 6 + 1 - \left(2^l \cdot 4 - 2^l \cdot \frac{24}{16}\right) \ &= \ 2^l \cdot \left(\frac{7}{2}\right) + 1. \end{aligned} \tag{19}$$

We also have for $j = 1, \ldots, d-1$

$$\begin{aligned} \sum_{i=0}^{j} m_i^{(1)} \geq (m_1 + p_1) + \sum_{i=2}^{j+1} m_i \ &= \ \sum_{i=0}^{j+1} m_i - (m_0 - p_1) \\ &\geq (j+2)2^l \cdot 3 + 1 - \left(2^l \cdot 4 - 2^l \cdot \frac{24}{16}\right) \\ &= 2^l \cdot \left(3(j+1) + \frac{1}{2}\right) + 1. \end{aligned} \tag{20}$$

Now we try to find non-trivial zero for $\mathscr{F}^{(1)}$. By the proof of Lemma 5.2 (see Remark 4) we can assume $2^l \cdot (\frac{7}{2}) + 1 \leq m_0^{(1)} \leq 2^l \cdot 4$. So we proceed as before and use the variables at level 0 to produce $2^{l-4} \cdot 28$ primaries at levels at least 1. We denote by $p_i^{(1)}$ the number of these variables that have level i. By the proof of Subcase A, we only have to work with $\sum_{i\geq 2} p_i^{(1)} < 2^{l-4} \cdot 4$. So we assume $p_1^{(1)} \geq 2^{l-4} \cdot 24$ and, applying the change of variables (18), we get an equivalent form $\mathscr{F}^{(2)}$ that satisfies

$$\begin{aligned} m_0^{(2)} \geq m_1^1 + p_1^1 &= m_0^1 + m_1^1 - (m_0^1 - p_1^1) \\ &\geq 2^l \cdot 6 + 1 + 2^l \cdot \left(\frac{1}{2}\right) - \left(2^l \cdot 4 - 2^l \cdot \frac{24}{16}\right) = 2^l \cdot 4 + 1 \end{aligned} \tag{21}$$

and

$$m_0^{(2)} + \cdots + m_j^{(2)} \geq (j+1)2^l \cdot 3 + 1.$$

By Lemma 5.2, we can find non-trivial zero for $\mathscr{F}^{(2)}$.

Case 2: $2^l \cdot 3 + 1 \leq m_0 \leq 2^{l-1} \cdot 7$.
We will follow the same lines as in Case 1. First we apply Lemma 3.4 to the variables at level 0 and get $2^{l-4} \cdot 24$ primaries at levels at least 1. Then we analyze two subcases.

Subcase A**:** $\sum_{i\geq 2} p_i \geq 2^{l-4} \cdot 8$.
Again we start by showing that we can assume $\sum_{i\geq 2} p_i \geq 2^{l-5} \cdot 32$. Indeed, if that is not the case, we would have

$$\sum_{i\geq 2} p_i = 2^{l-4} \cdot 8 + \epsilon \text{ with } 0 \leq \epsilon < 2^{l-5} \cdot 16,$$

which would imply $p_1 \geq 2^{l-4} \cdot 16 - \epsilon \geq 2(2^{l-5} \cdot 16 - \epsilon)$. We have $m_1 \geq 2^{l-1} \cdot 5 + 1$ (see (14)) and then we use Corollary 3.3 to contract these variables and get $s_j \geq 1$ for $j = 1, 2, \ldots, l-1$. Applying Lemma 3.6 to the (1)-primaries and one (1)-secondary we get $2^{l-5} \cdot 16 - \epsilon$ new primaries at levels at least 2 which gives us

$$\sum_{i\geq 2} p_i \geq 2^{l-5} \cdot 16 + \epsilon + 2^{l-5} \cdot 16 - \epsilon = 2^{l-5} \cdot 32.$$

Now we can proceed exactly as in Subcase A of Case 1. Indeed, here we have many more secondaries (due to the smaller bound on m_0) so even though we have used more (1)-secondaries, this is balanced by the extra secondaries we have now and every step we did there can also be done here.

Subcase B: $\sum_{i\geq 2} p_i < 2^{l-4}\cdot 8$.
Here we have $p_1 \geq 2^{l-4}\cdot 16+1$. We apply the change of variables (18) and get an equivalent form $\mathscr{F}^{(1)}$ satisfying

$$m_0^{(1)} \;\geq\; m_1+p_1 \;=\; m_0+m_1-(m_0-p_1) \;=\; 2^l\cdot\binom{7}{2}+2 \tag{22}$$

and

$$m_0^{(1)}+\cdots+m_j^{(1)} \geq (j+1)2^l\cdot 3+1$$

for $j=1,\ldots,d-1$. But then by Case 1 we can find non-trivial zero for $\mathscr{F}^{(1)}$. □

Proposition 5.4. *Let K be the unramified quadratic extension of $\mathbb{Q}_2$. Any normalized form $\mathscr{F}$ of degree $d=2^4\cdot m$ with $m\geq 3$ odd in $N=d^2+1$ variables has non-trivial zeros over K.*

Proof. By Lemma 5.2 we can assume $2^4\cdot 3+1=49\leq m_0\leq 64=2^4\cdot 4$. Our goal here is to create a (6)-primary variable.

Case 1: $63\leq m_0\leq 64$.
We use Lemma 3.4 and contract the variables at level 0 to create 31 primaries at levels at least 1. That is,

$$31=p_1+p_2+\cdots+p_5.$$

We now show that if $p_2+\cdots+p_5\geq 16$ then we can create a (6)-primary. Since $m_0\leq 64$ we have $m_1\geq 33$ (see (14)) and then we apply Corollary 3.3 in order to get $s_1,s_2,s_3\geq 1$. Note that these contractions use at most 19 variables. We can assume $p_4+p_5\geq 4$. Indeed, supposing otherwise we have

$$p_4+p_5=\epsilon \text{ and } p_2+p_3\geq 16-\epsilon\geq 4(4-\epsilon),\quad 0\leq\epsilon\leq 3.$$

Applying Lemma 3.6 to the variables at levels 2 and 3 we get $4-\epsilon$ new primaries at levels at least 4 and then we have $p_4+p_5\geq\epsilon+4-\epsilon=4$. If $m_4\geq 7$ then we are done, since we can use Lemma 3.2 to obtain $s_4,s_5\geq 1$ and then create a (6)-primary using Lemma 3.6. But if $m_4\leq 6$, we get $m_1+\cdots+m_3\geq 171$ (see (14)) and since at most 19 variables were used in the contractions involving the (1)-secondaries we still have at least 152 variables at our disposal. Then we can use these variables to create more than seven (4)-secondaries applying Corollary 3.3.

So we can assume $p_1 \geq 16$. If $p_1 = 16$, we use these variables and one (1)-secondary to get eight new primaries at higher levels (Lemma 3.6) and then $p_2 + \cdots + p_5 \geq 23 \geq 16$ so we can proceed as we just did above. This same argument shows that we only have to analyze the case $p_1 = 31$. In this case we apply the change of variables (18) and get an equivalent form $\mathscr{F}^{(1)}$ satisfying

$$m_0^{(1)} \geq 4 \cdot 2^4$$

and $m_0^{(1)} + \cdots + m_j^{(1)} \geq (3j+4) \cdot 2^4$ for $j = 1, 2, \ldots, d-1$. It is sufficient to find a non-trivial zero for $\mathscr{F}^{(1)}$. By the proof of Lemma 5.2 we can assume $m_0^{(1)} = 64$. Repeating the argument we just gave, we obtain an equivalent form $\mathscr{F}^{(2)}$ for which

$$m_0^{(2)} \geq 5 \cdot 2^4 - 1$$

and $m_0^{(2)} + \cdots + m_j^{(2)} \geq (3j+5) \cdot 2^4 - 1$ for $j = 1, \cdots, d-2$. The proof of Lemma 5.2 shows that the form $\mathscr{F}^{(2)}$ has a non-trivial zero.

Case 2: $61 \leq m_0 \leq 62$.
We use Lemma 3.4 to contract the variables at level 0 and get 30 primaries at levels at least 1. Just as in Case 1, if $p_2 + \cdots + p_5 \geq 16$ we can get a (6)-primary since here we have even more secondaries due to the smaller bound on m_0. So the reasoning used in Case 1 shows that we can assume $p_1 \geq 29$. We apply the change of variables (18) to create an equivalent form $\mathscr{F}^{(1)}$ for which

$$m_0^{(1)} \geq 4 \cdot 2^4$$

and $m_0^{(1)} + \cdots + m_j^{(1)} \geq (3j+4) \cdot 2^4$ for $j = 1, \ldots, d-1$. By Case 1 we know that the form $\mathscr{F}^{(1)}$ has a non-trivial zero.

Case 3: $57 \leq m_0 \leq 60$.
We contract the variables at level 0 and get 28 primaries at levels at least 1 (Lemma 3.4). The same argument used in Case 1 shows that we can assume $p_1 \geq 25$. We apply the change of variables (18) to create an equivalent form $\mathscr{F}^{(1)}$ for which

$$m_0^{(1)} \geq 4 \cdot 2^4 - 2$$

and $m_0^{(1)} + \cdots + m_j^{(1)} \geq (3j+4)2^4 - 2$ for $j = 1, \ldots, d-1$. By Case 2 we know that $\mathscr{F}^{(1)}$ has a non-trivial zero.

Case 4: $49 \leq m_0 \leq 56$.
We contract the variables at level 0 and get 24 primaries at levels at least 1 (Lemma 3.4). The same argument used in Case 1 shows that we can assume $p_1 \geq 17$. We apply the change of variables (18) to create an equivalent form $\mathscr{F}^{(1)}$ for which

$$m_0^{(1)} \geq 4 \cdot 2^4 - 6$$

and $m_0^{(1)} + \cdots + m_j^{(1)} \geq (3j+4)2^4 - 6$ for $j = 1, \ldots, d-1$. By Case 3 we know that $\mathscr{F}^{(1)}$ has a non-trivial zero.

□

Proposition 5.5. *Let K be the unramified quadratic extension of $\mathbb{Q}_2$. Any normalized form $\mathscr{F}$ of degree $d = 2^3 \cdot m$ with $m \geq 3$ odd, in $N = d^2+1$ variables has non-trivial zeros over K.*

Proof. Our goal now is to obtain a (5)-primary variable. By Lemma 5.2 we only need to analyze the cases $2^3 \cdot 3 + 1 = 25 \leq m_0 \leq 32 = 2^3 \cdot 4$.

Case 1: $31 \leq m_0 \leq 32$.
We use Lemma 3.4 and contract the variables at level 0 to 15 primaries at levels at least 1.

We assert that if $p_2 + p_3 + p_4 \geq 8$, then we can create a (5)-primary. Since $m_0 \leq 32$ we have $m_1 \geq 17$ (see (14)) and by Lemma 3.2 we can contract two of these variables and get $s_2 \geq 1$. We can assume $p_3 + p_4 \geq 4$. Indeed, supposing otherwise we have

$$p_3 + p_4 = \epsilon \text{ and } p_2 \geq 8 - \epsilon \geq 2(4-\epsilon), \quad 0 \leq \epsilon \leq 3.$$

Then applying Lemma 3.6 to the variables at level 2 we get $4 - \epsilon$ new primaries at levels at least 3 and we have $p_3 + p_4 \geq 4$. If $m_3 \geq 7$ we can obtain $s_3, s_4 \geq 1$ (Lemma 3.2), and then Lemma 3.6 gives us a (5)-primary. So we can assume $m_3 \leq 6$. In this case we have (see (14))

$$m_1 + m_2 \geq 59.$$

Although we have already used a few variables from level 1, in the worst case we are still left with at least 56 of these variables, and we can use them to obtain

$s_3 \geq 7$ (Corollary 3.3) and proceed as before.

Hence we can assume $p_1 \geq 8$. If $p_1 = 8$, then we use one (1)-secondary and the (1)-primaries to get four new primaries at levels at least 2 (Lemma 3.6). This implies $p_2 + p_3 + p_4 \geq 7 + 4 = 11 > 8$ and we can deal with this case as above. Following this reasoning we conclude that the only delicate case is $p_1 = 15$. In this case we apply the change of variables (18) and get an equivalent form $\mathscr{F}^{(1)}$ that satisfies

$$m_0^{(1)} \geq 4 \cdot 2^3$$

and $m_0^{(1)} + \cdots + m_j^{(1)} \geq (3j+4) \cdot 2^3$ for $j = 1, \ldots, d-1$. It is sufficient to find a non-trivial zero for $\mathscr{F}^{(1)}$. By the proof of Lemma 5.2 we can assume $m_0^{(1)} = 4 \cdot 2^3$ and use the same approach as above. That is, we use the variables at level 0 to create 15 primaries at levels at least 1. Again we can assume $p_1 = 15$ and then applying the change of variables (18) we get a second equivalent form $\mathscr{F}^{(2)}$ satisfying

$$m_0^{(2)} \geq 39 > 2^3 \cdot 4 + 1$$

and $m_0^{(2)} + \cdots + m_j^{(2)} \geq (3j+5) \cdot 2^3 - 1$ for $j = 1, \ldots, d-2$. We can obtain a non-trivial zero for this form by the proof of Lemma 5.2.

Case 2: $29 \leq m_0 \leq 30$.
We contract the variables at level 0 to 14 primaries at levels at least 1 (Lemma 3.4). Just as in Case 1, if $p_2 + p_3 + p_4 \geq 8$ we can get a (5)-primary since here we have even more secondaries due to the smaller bound on m_0. So the same reasoning used in Case 1 shows that we can assume $p_1 \geq 13$. We apply the change of variables (18) and create an equivalent form $\mathscr{F}^{(1)}$ satisfying

$$m_0^{(1)} \geq 32$$

and $m_0^{(1)} + \cdots + m_j^{(1)} \geq (3j+4) \cdot 2^3$ for $j = 1, \ldots, d-1$. By Case 1 we can find a non-trivial zero for this form.

Case 3: $25 \leq m_0 \leq 28$.
We contract the variables at level 0 to 12 primaries at levels at least 1. The same reasoning used in Case 1 shows that we can assume $p_1 \geq 9$. We apply the change of variables (18) and create an equivalent form $\mathscr{F}^{(1)}$ satisfying

$$m_0^{(1)} \geq 30$$

and $m_0^{(1)} + \cdots + m_j^{(1)} \geq (3j+4) \cdot 2^3 - 2$ for $j = 1, \ldots, d-1$. By Case 2 we can find a non-trivial zero for this form. □

Proposition 5.6. *Let K be the unramified quadratic extension of $\mathbb{Q}_2$. Any normalized form $\mathscr{F}$ of degree $d = 2^2 \cdot m$ with $m \geq 3$ odd, in $N = d^2 + 1$ variables has non-trivial zeros over K.*

Proof. Now we have to create a (4)-primary. By Lemma 5.2 we only need to analyze the cases $2^2 \cdot 3 + 1 = 13 \leq m_0 \leq 16 = 2^2 \cdot 4$.

Case 1: $15 \leq m_0 \leq 16$.
We apply Lemma 3.4 to the variables at level 0 and obtain $p_1 + p_2 + p_3 \geq 7$.

We assert that if $p_2 + p_3 \geq 4$, then we can create a (4)-primary. Since $m_0 \leq 16$ we have $m_1 \geq 9$ (see (14)) and Lemma 3.2 gives us $s_1, s_2 \geq 1$. We can assume $p_3 \geq 2$ (otherwise apply Lemma 3.6). If $m_3 \geq 1$ we are done by Lemma 3.6. If not we have $m_1 + m_2 \geq 33$ (see (14)). Of these variables, there are still at least 30 remaining after our previous contractions, and we can use them to create a (3)-secondary by Corollary 3.3.

Then we can assume $p_1 \geq 4$. If $4 \leq p_1 \leq 6$ then we can use Lemma 3.4 or 3.6 to contract the (1)-primaries to get $p_2 + p_3 \geq 4$ (remember that $m_1 \geq 9$) and then proceed as above. Even though we may have used an extra secondary in an appeal to Lemma 3.6, we still have enough variables to carry out the argument. So we assume $p_1 = 7$. We apply the change of variables (18) to create an equivalent form $\mathscr{F}^1$ satisfying

$$m_0^1 \geq 16$$

and $m_0^1 + \cdots + m_j^1 \geq (3j + 4) \cdot 2^2$ for $j = 1, 2, \ldots, d - 1$. It is sufficient to find non-trivial zero for $\mathscr{F}^1$. By the previous cases we only need to analyze the case $m_0^1 = 16$. Proceeding exactly as above, we get a second equivalent form $\mathscr{F}^2$ satisfying

$$m_0^2 \geq 19$$

and $m_0^2 + \cdots + m_j^2 \geq (3j + 5) \cdot 2^2 - 1$ for $j = 1, 2, \ldots, d - 2$. By the previous cases this form has a non-trivial zero.

Case 2: $13 \leq m_0 \leq 14$.
We contract the variables at level 0 and get $p_1 + p_2 + p_3 = 6$ (Lemma 3.4). Just as in Case 5, if $p_2 + p_3 \geq 4$ we can get a (4)-primary (in fact here we have more secondaries due to the smaller bound on m_0). So we may assume $p_1 \geq 3$. In fact, we can assume $p_1 \geq 5$, for if that is not the case we can apply Lemma 3.6 to obtain $p_2 + p_3 \geq 4$ and then proceed as above (since $m_0 \leq 14$ we have $m_1 \geq 11$). So

we assume $p_1 \geq 5$ and make the change of variables (18) to create an equivalent form $\mathscr{F}^1$ satisfying

$$m_0^1 \geq 15$$

and $m_0^1 + \cdots + m_j^1 \geq (3j+4)2^2 - 1$ for $j = 1, 2, \ldots, d-1$. By our previous cases we know that $\mathscr{F}^1$ has a non-trivial zero.

□

Proposition 5.7. *Let K be the unramified quadratic extension of $\mathbb{Q}_2$. Any normalized form $\mathscr{F}$ of degree $d = 2 \cdot m$ with $m \geq 3$ odd, in $N = d^2 + 1$ variables has non-trivial zeros over K.*

Proof. Now we need to construct a (3)-primary. Again we divide the proof into cases according to the value of $m_0 \geq 7$.

Case 1: $m_0 \geq 15$.
The result follows directly from Corollary 3.5.

Case 2: $11 \leq m_0 \leq 14$.
Here we analyze two subcases.

Subcase A**:** $m_2 \geq 1$.
We apply Lemma 3.4 to the variables at level 0 and get $p_2 \geq 2$. Then we use Lemma 3.6 to create a (3)-primary.

Subcase B**:** $m_2 = 0$.
In this case we have $m_1 \geq 5$ (see (14)). If at least three of these (1)-secondaries have the same zeroterm, we use Lemma 3.1(c) to create a (2)-secondary and proceed as in Subcase A. Assume then that at most two of these five (1)-secondaries have equal zeroterms. Then we can find two pairs of (1)-secondaries such that the variables in each pair have distinct zeroterms (see (2) and (3)). Applying Lemma 3.4 to the variables at level 0 we get $p_1 + p_2 \geq 5$. If $p_2 \geq 3$ we are done by Lemma 3.4. If $p_2 = 1$ or 2, then $p_1 \geq 3$ and each pair of (1)-secondaries with distinct zeroterms together with one (1)-primary can be used to produce a (2)-primary (Lemma 3.1(b)). Then we have $p_2 \geq 3$ and can proceed as before. Finally, if $p_3 = 0$ then each pair of (1)-secondaries with distinct zeroterms together with one (1)-primary can be used to produce a (2)-primary (Lemma 3.1(b)), and with the remaining three (1)-primaries we construct a third (2)-primary using Lemma 3.4. We then proceed as before.

Case 3: $9 \leq m_0 \leq 10$.
Here we have $m_1 \geq 3$ (see (14)). We contract the variables at level 0 and get $p_1 + p_2 \geq 4$ (Lemma 3.4). Then Lemma 3.6 gives us $p_2 \geq 2$. If $m_2 \geq 1$, we are done by Lemma 3.6. If not, we must have $m_1 \geq 9$. At most one of these variables has been used (in our appeal to Lemma 3.6), and by Lemma 3.2 we can use the remaining eight to create a (2)-secondary.

Case 4: $7 \leq m_0 \leq 8$.
We apply Lemma 3.4 to the variables at level 0 and get $p_1 + p_2 \geq 3$.

First assume $p_2 \geq 1$. Since $m_0 \leq 8$ we have $m_1 \geq 5$ (see (14)). If three of these (1)-secondaries have the same zeroterm, then we can contract two of them and have $s_i \geq 1$ for $i = 1, 2$ (Lemma 3.1(c)). Then we can assume $p_2 \geq 2$ (otherwise apply Lemma 3.6 to the variables at level 1) and we can create a (3)-primary using Lemma 3.6. On the other hand, if we can not choose three (1)-secondaries with equal zeroterms, there must be two pairs of (1)-secondaries such that in each pair the variables have distinct zeroterms (see (2) and (3)). Each of these pairs, together with one (1)-primary, can be contracted to a (2)-primary (Lemma 3.1(b)) and this construction allows us to get $p_2 \geq 3$. Then we can create a (3)-primary by Lemma 3.4.

Then we can assume $p_1 \geq 3$. We again appeal to the change of variables (18) to get an equivalent form $\mathscr{F}^{(1)}$ satisfying

$$m_0^{(1)} \geq 8$$

and $m_0^{(1)} + \cdots + m_j^{(1)} \geq (j+1)2 \cdot 3 + 2$, for $j = 1, \ldots, d-1$. It is sufficient to find a non-trivial zero for $\mathscr{F}^{(1)}$. By our previous cases we can assume $m_0^{(1)} = 8$. Repeating the arguments we just made, we obtain a second equivalent form $\mathscr{F}^{(2)}$ satisfying

$$m_0^{(2)} \geq 9$$

and $m_0^{(2)} + \cdots + m_j^{(2)} \geq (j+1)2 \cdot 3 + 3$, for $j = 1, \ldots, d-2$. By our previous cases the form $\mathscr{F}^{(2)}$ admits non-trivial zero.

□

Now the proof of Theorem 1 for K the unramified quadratic extension of $\mathbb{Q}_2$ follows directly from Propositions 5.1, 5.3, 5.4, 5.5, 5.6 and 5.7.

Proof of theorem 1 for ramified quadratic extensions of $\mathbb{Q}_2$

We are assuming $\mathscr{F}$ to have all the properties described at the beginning of section 5. Our goal is to obtain a primary variable at level at least $2l+3$, for this suffices to prove Theorem 1 when K is ramified, according to Remark 1.

Lemma 5.8. *Let K be a ramified quadratic extension of $\mathbb{Q}_2$ and $\mathscr{F}$ be a normalized form of degree $d = 2^l \cdot m$ with $l \geq 1$, $m \geq 3$ odd, in $N = d^2+1$ variables. If $m_0 \geq 2^l \cdot 6 - 3$ then $\mathscr{F}$ has non-trivial zeros over K.*

Proof. Let us assume $m_0 \geq 2^{l+1} \cdot 4 - 3$. By Corollary 4.3(a), we can obtain $p_{2l+2} \geq 1$ and another primary at level at least $2l+2$. We may assume $p_{2l+2} \geq 2$, otherwise we are done. But this is enough to get a primary at level $2l+3$ or higher by Lemma 4.1(a).

Now assume $2^l \cdot 7 - 3 \leq m_0 \leq 2^l \cdot 8 - 4$. We apply Corollary 4.3 to the variables at level 0 and get four $(2l)$-primaries and another primary at level at least $2l$ and at most $2l+2$ (otherwise we are done). If the level of this last primary is higher than $2l$ we contract two pairs of the four $(2l)$-primaries and get two more primaries at levels higher than $2l$ (Lemma 4.1(a)). Then the result follows from Lemma 4.8. Now if $p_{2l} \geq 5$ we get $p_{2l+2} \geq 2$ (Lemma 4.1(b)) and these two variables can be contracted to a $(2l+3)$-primary by Lemma 4.1(a).

Finally, assume $2^l \cdot 6 - 3 \leq m_0 \leq 2^l \cdot 7 - 4$. We can apply Corollary 4.3 to get three primaries at level $2l$ and another primary at level at least $2l$. If $p_{2l+2} \geq 1$ we can contract the three $(2l)$-primaries to obtain $p_{2l+2} \geq 2$, by Lemma 4.1(b), and complete the proof by applying Lemma 4.1(a). Hence we may assume that either $p_{2l} \geq 4$ or else that $p_{2l} \geq 3$ and $p_{2l+1} \geq 1$. In both cases we can use Lemma 4.1(a) in order to get two primaries at levels at least $2l+1$ and at most $2l+2$. If $m_{2l+1} \geq 1$ or $m_{2l+2} \geq 1$ we would get a $(2l+3)$-primary by applying Lemma 4.8. So we assume $m_{2l+1} = m_{2l+2} = 0$, and it follows from (14) that

$$\begin{aligned} m_1 + \cdots + m_{2l} &\geq [(2l+2)+1]2^l \cdot 3 + 1 - m_0 - m_{2l+1} - m_{2l+2} \\ &\geq (3l+1)2^{l+1} + 5. \end{aligned}$$

Since at least $(3l+1)2^l + 3$ of these secondary variables are at levels which have the same parity, it follows from Corollary 4.3(b) that we can get either $s_{2l+2} \geq 1$ or $s_{2l+1} \geq 1$ and complete the proof by applying Lemma 4.8. □

Proposition 5.9. *Let K be a ramified quadratic extension of $\mathbb{Q}_2$. Any normalized form $\mathscr{F}$ of degree $d = 2 \cdot m$ with $m \geq 3$ odd, in $N = d^2+1$ variables has non-trivial zeros over K.*

Proof. By Remark 1 it is enough to produce a primary at level at least 5. It follows from (14) and Lemma 5.8 that we may assume

$$7 \leq m_0 \leq 8.$$

By Lemma 4.2 we obtain $p_2 \geq 2$ and another primary at level at least 2 and at most 4 (otherwise we are done). Suppose this extra primary is at level 3 or 4. By Lemma 4.1(a) applied to $p_2 \geq 2$ we obtain another primary at level 3 or 4. If either $m_3 \neq 0$ or $m_4 \neq 0$ we can apply Lemma 4.8 to conclude the proof. If $m_3 = m_4 = 0$ then (see (14))

$$m_1 + m_2 \geq 31 - m_0 - m_3 - m_4 \geq 31 - 8 = 23.$$

Thus either $m_1 \geq 12$ or $m_2 \geq 12$. In any case, Lemma 4.2 gives either $s_3 \neq 0$ or $s_4 \neq 0$, and we can proceed as above. Hence the extra primary is at level 2 and we must have $p_2 \geq 3$. If $m_2 \geq 3$, then since $p_2 \geq 3$ we apply Lemma 4.1(a) and get three primaries at levels 3 or 4 and then Lemma 4.8 gives us a (5)-primary. If $m_2 \leq 2$, then we can apply Lemma 4.1(b) to the three primaries at level 2 to obtain $p_4 \geq 1$. If $m_4 \neq 0$ the result follows from Lemma 4.1(a). Hence we may assume $m_2 \leq 2$ and $m_4 = 0$. By (14) we have

$$m_1 \geq 19 - m_0 - m_2 \geq 19 - 8 - 2 = 9.$$

Now we turn our attention to B-primaries, and according to Remark 2, to conclude the proof it is sufficient to produce a B-primary at level at least 6. Let us use the notation $p_k(B)$ to represent the number of (k)-B-primaries. Since $m_1 \geq 9$, it follows from Lemma 4.2 that $p_3(B) \geq 3$. Using Lemma 4.1(b) on these B-primaries, we get $p_5(B) \geq 1$ (otherwise we are done). Therefore we must have $m_5 = 0$, as otherwise Lemma 4.1(a) would give us the result. If $m_3 \geq 5$ we could apply Lemma 4.1(c) to get $s_5 \neq 0$, and the result would follow. Therefore we may assume

$$m_2 \leq 2, \quad m_3 \leq 4, \quad m_4 = 0, \quad \text{and} \quad m_5 = 0.$$

Then we have (see (14))

$$m_1 \geq 37 - m_0 - m_2 - m_3 - m_4 - m_5 \geq 37 - 8 - 2 - 4 = 23.$$

By Corollary 4.3 applied to m_1 we can get $p_5(B) \geq 3$, which is enough to produce a (B)-primary at level at least 6 by Lemma 4.1(a), concluding the proof. □

Lemma 5.10. *Let K be a ramified quadratic extension of $\mathbb{Q}_2$ and $\mathscr{F}$ be a normalized form of degree $d = 2^l \cdot m$ with $m \geq 3$ odd, in $N = d^2+1$ variables. If $l \geq 2$ and $m_0 \geq 2^l \cdot 5 - 3$, then $\mathscr{F}$ has non-trivial zeros over K.*

Proof. By Lemma 5.8 we may assume $m_0 \leq 2^l \cdot 6 - 4$. Using Corollary 4.3 we get seven $(2l-2)$-primaries and an eighth primary at level at least $2l-2$. (Naturally we assume the level of this last variable is less than $2l+3$.) Regardless of the level of this eighth primary, we can use Lemma 4.1 to get two primary variables at level $2l$, one at a level at least $2l$ and a fourth at a level at least $2l-1$. We analyze two subcases.

Subcase A**:** $p_{2l+1} + p_{2l+2} \geq 1$.
Here we apply Lemma 4.1(a) to the two primaries at level $2l$ and get a second primary at level $2l+1$ or $2l+2$. Then if $m_{2l+1} \geq 1$ or $m_{2l+2} \geq 1$ we can get a $(2l+3)$-primary by applying Lemma 4.8. So we assume $m_{2l+1} = m_{2l+2} = 0$. Now we have (see (14))

$$\begin{aligned} m_1 + \cdots + m_{2l} &\geq (2l+3)2^l \cdot 3 + 1 - m_0 - m_{2l+1} - m_{2l+2} \\ &\geq 2^l \cdot (6l+3) + 5 \geq 2^l \cdot 15 + 5. \end{aligned}$$

At least $2^{l-1} \cdot 15 + 3$ of these variables are at levels of equal parity so we can apply Corollary 4.3 and get $s_{2l+1} \geq 1$ or $s_{2l+2} \geq 1$, so we can repeat the arguments above and get the result.

Subcase B**:** $p_{2l+1} + p_{2l+2} = 0$.
If $p_{2l} \geq 4$ we can apply Lemma 4.1(b) to get $p_{2l+2} \geq 1$ and $p_{2l} \geq 2$ and we can proceed as in Subcase A. Thus assume $p_{2l} = 3$ and $p_{2l-1} = 1$. If $m_{2l-1} \neq 0$, then this fact, along with $p_{2l-1} = 1$, gives us a fourth primary at level at least $2l$ (Lemma 4.1(a)) and this case has already been treated (The one secondary variable we have used here will not make a difference in the argument.) If $m_{2l} \geq 3$ then since $p_{2l} = 3$, we can get three primaries at levels at least $2l+1$, which is enough to get a primary at level at least $2l+3$, according to Lemma 4.8. Hence we may assume $m_{2l-1} = 0$ and $m_{2l} \leq 2$. Then we have (see (14))

$$m_1 + \cdots + m_{2l-2} \geq (2l+1)2^l \cdot 3 + 1 - m_0 - m_{2l-1} - m_{2l} \geq (6l-3) \cdot 2^l + 3$$

and at least $(6l-3) \cdot 2^{l-1} + 1$ of these variables are at levels of equal parity so we can apply Corollary 4.3 and get either $s_{2l-1} \geq 6$ or $s_{2l} \geq 6$. Now we can proceed as we did when $m_{2l-1} \neq 0$ or $m_{2l} \geq 3$. □

Proposition 5.11. *Let K be a ramified quadratic extension of $\mathbb{Q}_2$. Any normalized form $\mathscr{F}$ of degree $d = 2^2 \cdot m$ with $m \geq 3$ odd, in $N = d^2 + 1$ variables has non-trivial zeros over K.*

Proof. By Lemma 5.10 and (14) we may assume $13 \leq m_0 \leq 16$. Now our goal is to produce a primary variable at level at least 7, by Remark 1. Applying Lemma 4.2, we get five (2)-primaries and another primary at level at least 2.

First assume $m_2 + m_4 \geq 5$. We assert that we can create two primaries at levels at least 5. Indeed, if $m_4 = 0$ we get $m_2 \geq 5$ and applying Lemma 4.2 we get $s_2 \geq 3$ and $s_4 \geq 1$ so Lemma 4.6 gives us the result. If $m_4 = 1$ we get the result again applying Lemma 4.6. If $m_4 \geq 2$ we apply Lemma 4.2 to the (2)-primaries and get two new primaries, one at level 4 and the other at level at least 4. These primaries can be contracted with the (4)-secondaries in order to create two primaries at levels at least 5 (Lemma 4.1(a)). Since in each case we can obtain these two primaries, we conclude that if we had $m_5 \neq 0$ or $m_6 \neq 0$ we would be done by Lemma 4.8. So we assume $m_5 = m_6 = 0$. But this implies (see (14)) $m_1 + \cdots + m_4 \geq 69$ and excluding the (possibly) five secondaries used earlier, we would have either $s_1 + s_3 \geq 32$ or $s_2 + s_4 \geq 32$. Applying Corollary 4.3, we get $s_5 \neq 0$ or $s_6 \neq 0$.

Thus let us assume $m_2 + m_4 \leq 4$. By (14) we have $m_1 \geq 37 - m_0 - m_2 \geq 17$. We now try to make a B-(8)-primary (see Remark 2). Corollary 4.3 gives us two B-primaries at level 5 and another B-primary at level at least 5. To get these variables we used at most 14 (1)-secondaries. We analyze two further subcases.

Subcase A: $m_3 + m_5 \geq 13$.
We apply Lemma 4.2 repeatedly and get $s_5 \geq 3$ and $s_7 \geq 1$. Using Lemma 4.1(a) we can contract the B-primaries and secondaries at level 5 to two new primaries at levels at least 6 and the result follows from Lemma 4.8.

Subcase B: $m_3 + m_5 \leq 12$.
Here we have (see (14))

$$m_1 \geq 6 \cdot 2^2 \cdot 3 + 1 - 4 - 12 - 16 = 41 > 2^3 \cdot 5$$

and applying Corollary 4.3 we get two B-(7)-primaries that can be contracted to a B-primary at level at least 8 by Lemma 4.1(a).

□

Proposition 5.12. *Let K be a ramified quadratic extension of $\mathbb{Q}_2$. Any normalized form $\mathscr{F}$ of degree $d = 2^l \cdot m$ variables has non-trivial zeros over K.*

PROOF. By (14) and Lemma 5.10 we may assume

$$2^l \cdot 3 + 1 \leq m_0 \leq 2^l \cdot 5 - 4.$$

We are going to divide this proof into many cases according to the value of m_0.
Case 1: $2^l \cdot (\frac{9}{2}) - 3 \leq m_0 \leq 2^l \cdot 5 - 4$.
We apply Corollary 4.3 and get six primaries at level $2l-2$ and another primary at level at least $2l-2$. We have (see (14))

$$m_1 + m_2 \geq (2^l \cdot 9 + 1) - m_0 \geq 2^l \cdot 4 + 5, \tag{23}$$

and we analyze two subcases.

Subcase A**.** Suppose $m_2 + m_4 + \cdots + m_{2l-2} \geq 2^l - 3$.
We either have $m_{2l-2} \geq 1$ or else we can apply Corollary 4.3(b) to the variables at levels $2, 4, \ldots, 2l-4$ and get $s_{2l-2} \geq 1$. We now have seven variables at level $2l-2$, six of them being primary. However, if we contract the secondary, it will be with a primary, so we can act as though all seven variables were primary. We still have an eighth primary at level at least $2l-2$. This situation has already been examined in the proof of Lemma 5.10. Here, the situation is even a bit better than it was there. While we have already used at most $2^l - 6$ variables during the initial contractions, our smaller bound on m_0 is more than enough to offset that loss. Hence, all the contractions we did there can still be done here, and we get the result.

Subcase B**.** Suppose $m_2 + m_4 + \cdots + m_{2l-2} \leq 2^l - 4$.
In particular $m_2 \leq 2^l - 4$, hence from (23) we have $m_1 \geq 2^l \cdot 3 + 9$ and we can apply Corollary 4.3 to these variables in order to get three B-primaries at level $2l-1$ and another B-primary at level at least $2l-1$. (We assume this level less than $2l+4$, as otherwise we would be done by Remark 2). We now have six A-primaries at level $2l-2$, three B-primaries at level $2l-1$, and a fourth B-primary at level at least $2l-1$. We analyze two further subcases.

First assume $m_{2l-1} \geq 13$. Applying Lemma 4.2 to these variables, we get a secondary variable at level $2l+3$. Then an application of Lemma 4.9 finishes the proof of this case.

Now suppose $m_{2l-1} \leq 12$. Then we have (see (14))

$$m_1 + \cdots + m_{2l-2} \geq ((2l)2^l \cdot 3 + 1) - m_0 - m_{2l-1} \geq 2^l \cdot (6l-5) - 7. \tag{24}$$

By the hypothesis of Subcase B, we must have

$$m_1 + m_3 + \cdots + m_{2l-3} \geq 2^l \cdot (6l - 6) - 3.$$

Subtracting the $2^l \cdot 3 - 3$ (1)-secondaries we contracted at the beginning of Subcase B from this amount, we still have

$$s_1 + s_3 + \cdots + s_{2l-3} \geq 2^l \cdot (6l - 9),$$

and we can then apply Corollary 4.3(b) to obtain $s_{2l-1} \geq 15$ (remember that we are assuming $l \geq 3$) and then we proceed as we did when $m_{2l-1} \geq 13$.

Case 2: $2^l \cdot 4 - 3 \leq m_0 \leq 2^l \cdot \left(\frac{9}{2}\right) - 4$.
We apply Corollary 4.3 to the variables at level 0 and get five primaries at level $2l - 2$ and another primary at level at least $2l - 2$. We have (see (14))

$$m_1 + m_2 \geq 9 \cdot 2^l + 1 - m_0 \geq 2^{l-1} \cdot 9 + 5. \tag{25}$$

We analyze two subcases.

Subcase A**:** Suppose $m_2 + m_4 + \cdots + m_{2l-2} \geq 2^l \cdot 2 - 3$.
We apply Corollary 4.3(b) and contract these variables, getting $s_{2l-2} \geq 3$ and $s_{2l} \geq 1$. So we have five $(2l - 2)$-primaries, three $(2l - 2)$-secondaries and one $(2l)$-secondary. By Lemma 4.6 we can obtain two primaries at levels higher than $2l$. If $m_{2l+1} \geq 1$ or $m_{2l+2} \geq 1$, together with these two primaries, we can use Lemma 4.8 and get the result. If $m_{2l+1} = m_{2l+2} = 0$, we have (see (14))

$$m_1 + \cdots + m_{2l} \geq (12l + 9)2^{l-1} + 5.$$

Extracting from this amount the $2^l \cdot 2 - 3$ secondaries we already used in this subcase, we still have $s_1 + \cdots + s_{2l} \geq (12l + 5)2^{l-1} + 8$, and we can apply Corollary 4.3 as before to either the variables at odd levels or those at even levels to get either a $(2l+1)$-secondary or a $(2l+2)$-secondary. Then we proceed as before.

Subcase B**:** Suppose $m_2 + m_4 + \cdots + m_{2l-2} \leq 2^l \cdot 2 - 4$.
By (25) we have $m_1 \geq 2^{l-1} \cdot 5 - 3$ and we can apply Corollary 4.3 to these variables in order to construct two B-primaries at level $(2l - 1)$ and another B-primary at level at least $2l - 1$. We analyze two further subcases.

First assume $m_{2l-1} \geq 13$. Applying Lemma 4.2(b) recursively to these variables we get $s_{2l+3} \geq 1$ and $s_{2l-1} \geq 3$. As before, we may treat one of these $(2l-1)$-secondaries as a B-primary (We will only allow contractions of this variable with B-primary variables). Thus we have five $(2l-2)$-A-primaries, a sixth A-primary at level at least $2l-2$, three $(2l-1)$-B-primaries, another B-primary at level at least $2l-1$, and one $(2l+3)$-secondary. By either Lemma 4.9 or Lemma 4.10 (depending on the level of the sixth A-primary), we can either get a $(2l+3)$-A-primary or a $(2l+4)$-B-primary and get the result.

Now assume $m_{2l-1} \leq 12$. Then we have (see (14))

$$m_1 + \cdots + m_{2l-2} \geq 6l \cdot 2^l + 1 - m_0 - m_{2l-1} \geq (12l-9)2^{l-1} - 7. \qquad (26)$$

By the hypothesis of Subcase B we must have

$$m_1 + m_3 + \cdots + m_{2l-3} \geq 2^{l-1} \cdot (12l - 13) - 3.$$

Subtracting the $2^{l-1} \cdot 5 - 3$ (1)-secondaries we have already used, we have

$$s_1 + s_3 + \cdots + s_{2l-3} \geq 2^{l-1} \cdot (12l - 18)$$

and we can use Corollary 4.3 to get $s_{2l-1} \geq 15$ (since we are assuming $l \geq 3$) and proceed as in the case $m_{2l-1} \geq 13$.

Case 3: $2^l \left(\frac{7}{2}\right) - 3 \leq m_0 \leq 2^l \cdot 4 - 4$.
Applying Corollary 4.3 to the variables at level 0, we get four primaries at level $2l-2$ and another primary at level at least $2l-2$. We have (see (14))

$$m_1 + m_2 \geq 9 \cdot 2^l + 1 - m_0 \geq 2^l \cdot 5 + 5. \qquad (27)$$

We analyze two subcases.

Subcase A: Suppose $m_2 + m_4 + \cdots + m_{2l-2} \geq 2^l \cdot 2 - 3$.
We apply Corollary 4.3(b) and get three $(2l-2)$-secondaries and one $(2l)$-secondary. These variables, together with the five primaries, can be contracted to two primaries at levels higher than $2l$ by Lemma 4.6. Now, if $m_{2l+1} \geq 1$ or $m_{2l+2} \geq 1$ then we are done by Lemma 4.8. But if $m_{2l+1} = m_{2l+2} = 0$ we have (see (14))

$$m_1 + \cdots + m_{2l} \geq (6l+5)2^l + 5.$$

Of these variables, at least $(6l+3)2^l+8$ variables are still unused, and we can apply Corollary 4.3 as before to variables at levels of the same parity and get either $s_{2l+1} \geq 1$ or $s_{2l+2} \geq 1$.

Subcase B: Suppose $m_2 + m_4 + \cdots + m_{2l-2} \leq 2^l \cdot 2 - 4$.
From (27) we have $m_1 \geq 2^l \cdot 3 - 3$. We apply Corollary 4.3 to these variables and get three B-primaries at level $2l-1$ and a fourth B-primary at level at least $2l-1$. We analyze two further subcases.

First suppose $m_{2l-1} \geq 13$. We apply Lemma 4.2(b) recursively to these variables and get $s_{2l-1} \geq 3$, $s_{2l+1} \geq 3$, and $s_{2l+3} \geq 1$. With three $(2l-1)$-B-primaries, another B-primary at level at least $2l-1$, three $(2l-1)$-secondaries, and two $(2l+1)$-secondaries, we apply Lemma 4.7 and get two B-primaries at levels higher than $2l+1$ (which we assume less than $2l+4$). These two B-primaries, together with one $(2l+3)$-secondary, give us one B-primary at level at least $2l+4$ by Lemma 4.8.

Now assume $m_{2l-1} \leq 12$. We have (see (14))

$$m_1 + \cdots + m_{2l-2} \geq 6l \cdot 2^l + 1 - m_0 - m_{2l-1} \geq (6l-4)2^l - 7. \tag{28}$$

By the hypothesis of this subcase we must have

$$m_1 + m_3 + \cdots + m_{2l-3} \geq 2^l \cdot (6l-6) - 3.$$

Subtracting the $2^l \cdot 3 - 3$ (1)-secondaries we have already used from this number, we still have

$$s_1 + s_3 + \cdots + s_{2l-3} \geq 2^l \cdot (6l-9).$$

We can now use Corollary 4.3(b) to get $s_{2l-1} \geq 15$ (we are assuming $l \geq 3$) and proceed as in the case $m_{2l-1} \geq 13$ above.

Case 4: $2^l \cdot 3 + 1 \leq m_0 \leq 2^l \cdot \left(\frac{7}{2}\right) - 4$.
We apply Corollary 4.3 to the variables at level 0 and get three primaries at level $(2l-2)$ and another primary at level at least $(2l-2)$. We have (see (14))

$$m_1 + m_2 \geq 9 \cdot 2^l + 1 - m_0 \geq 2^{l-1} \cdot 11 + 5. \tag{29}$$

We analyze two subcases.

Subcase A**:** Suppose $m_2 + m_4 + \cdots + m_{2l-2} \geq 2^{l-1} \cdot 5 - 3$.
We apply Corollary 4.3(b) and obtain $s_{2l-2} \geq 3$ and $s_{2l} \geq 2$. Then applying Lemma 4.7 we get two primaries at levels higher than $2l$. If $m_{2l+1} \geq 1$ or $m_{2l+2} \geq 1$, we are done by Lemma 4.8. But if $m_{2l+1} = m_{2l+2} = 0$ we have (see (14))

$$m_1 + \cdots + m_{2l} \geq (12l + 11)2^{l-1} + 5.$$

Of these variables, at least $(12l+6)2^{l-1}+8$ have not yet been used, and as before we can apply Corollary 4.3 to variables at levels of the same parity and get either $s_{2l+1} \geq 1$ or $s_{2l+2} \geq 1$, as needed.

Subcase B**:** Suppose $m_2 + m_4 + \cdots + m_{2l-2} \leq 2^{l-1} \cdot 5 - 4$.
This implies $m_1 \geq 2^l \cdot 3 + 9$ (see (29)) and we can proceed exactly as in Case 3, Subcase B. Again, while the exact numbers are different, we have enough secondary variables to make all the needed contractions. This completes the proof of the proposition. □

Now the proof of Theorem 1 for K being any quadratic ramified extension of $\mathbb{Q}_2$ follows directly from Propositions 5.1, 5.9, 5.11 and 5.12.

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